SOLUTIONS to the previous problems

1. $f(x) = \dfrac{x^4 - x^2 + 2}{3x^4 + x^2 + 5}$

$$\lim_{x \to \pm\infty} \frac{x^4 - x^2 + 2}{3x^4 + x^2 + 5} = \lim_{x \to \pm\infty} \frac{1 - \frac{1}{x^2} + \frac{2}{x^4}}{3 + \frac{1}{x^2} + \frac{5}{x^4}} = \frac{1}{3}$$

2. $f(x) = \dfrac{2x^5 - 2x^3 + 18}{x^4 + x^2 - x + 2}$

   SOLUTION: As x goes to infinity (positive or negative), this function will behave like $2x$, and so it will also go to $\pm\infty$. Therefore, there are no horizontal asymptotes.

   SIDE REMARK: Students in pre-calculus may recognize $y = 2x$ as a *slant asymptote*, but we won't discuss that here.

3. $f(x) = \dfrac{2x^5 - 2x^3 + 18}{x^4 + 3x^3 - x + 2} - 2x$

   SOLUTION: From the previous exercise, we might expect that this limit will probably be zero. However, we know that $\infty - \infty$ may not be zero, so we'll need to do more work to see what happens: Make this a single fraction, then divide numerator and denominator by x^n (for some n). First the algebra:

$$\frac{2x^5 - 2x^3 + 18}{x^4 + 3x^3 - x + 2} - 2x = \frac{2x^5 - 2x^3 + 18 - (2x)(x^4 + 3x^3 - x + 2)}{x^4 + 3x^3 + 2} =$$

$$\frac{-6x^4 - 2x^3 + 2x^2 - 4x + 18}{x^4 + 3x^3 + 2}$$

   It is now easy to see that the limit is -6!

4. $f(x) = \sin(x)$

   SOLUTION: Since the sine oscillates, it never "settles down" to any single number (Therefore, there is no horizontal asymptote).

5. $f(x) = \dfrac{\cos(x)}{\ln(\ln(x))}$

   SOLUTION: The numerator bounces around between ± 1, but the denominator increases without bound- The entire fraction then will go to zero as $x \to \infty$ (there is no horizontal asymptote at $-\infty$ since negative numbers [and zero] are not in the domain).

Find the limit:

1. $\lim_{x\to\infty} \tan^{-1}(x^2 + \sin(x) + e^{\sqrt{x}})$

   SOLUTION: We need two "facts": First, since $\tan(x)$ has vertical asymptotes at $x = \pm\pi/2$, the inverse tangent (or arctangent) has horizontal asymptotes at $y = \pm\pi/2$.

   Secondly, as $x \to \infty$, the function $x^2 + \sin(x) + \mathrm{e}^{\sqrt{x}}$ increases without bound.

   Therefore, as the input to the inverse tangent increases without bound, the function values get closer and closer to $y = \pi/2$.

2. $\lim_{x\to\infty} (\mathrm{e}^{-x} + 2\cos(3x))$

   SOLUTION: The limit does not exist. The first function, e^{-x} goes to zero, but $2\cos(3x)$ oscillates between ± 2 and never settles down to a single value.

3. $\lim_{x\to-\infty} (\sqrt{x^2 + x + 1} + x)$

   SOLUTION: This is an "$\infty - \infty$" kind of problem.

$$\lim_{x\to-\infty} \sqrt{x^2+x+1} + x \cdot \frac{\sqrt{x^2+x+1} - x}{\sqrt{x^2+x+1} - x} = \lim_{x\to-\infty} \frac{x^2+x+1-x^2}{\sqrt{x^2+x+1} - x}$$

   Divide numerator and denominator by x (or $x = -\sqrt{x^2}$, as needed in the denominator):

$$\lim_{x\to-\infty} \frac{x+1}{\sqrt{x^2+x+1} - x} = \lim_{x\to-\infty} \frac{1 + \frac{1}{x}}{-\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} - 1} = -\frac{1}{2}$$