Math 125: Exam 3 Review

Since we're using calculators, to keep the playing field level between all students, I will ask that you refrain from using certain features of your calculator, including graphing. Here is the statement that will appear on the exam and that I will ask that you sign:

> *I have not used my calculator on this examination except for arithmetic, trigonometric, logarithmic, and exponential functions. I certify that the work on this exam is my own and that I have not discussed the specific contents of this exam with anyone prior to taking it.*

A Quick Overview:

In this portion of the course, we've been focusing on applications of the derivative, Sections 3.7-4.3 with Section 4.8 added in.

1. Rates of Change (3.7)

   In terms of physics, know that the derivative of displacement is velocity, and the second derivative is acceleration (like exercises 1-10).

   In terms of economics, the marginal cost function is the derivative of the cost function, and $C'(x)$ is used as an approximation to the (additional) cost of producing $x+1$ items:

   $$C'(x) \approx \frac{C(x+1)-C(x)}{x+1-x} = C(x+1)-C(x)$$

2. Exponential Growth and Decay model:

   The growth and decay model is $y' = ry$. The function that satisfies this differential equation is: $y(t) = P_0 \mathrm{e}^{rt}$ (or "Pert").

   Be able to find half life or doubling time, or use information about the rate of change to find r and perhaps P_0. Be able to apply the model to radioactive decay, bacterial growth, or growth of a financial investment (with continuous compounding).

3. Related Rates (3.9)

   Formulas we should be familiar with: Area, perimeter, circumference. We should also know Pythagorean Theorem and Similar Triangles.

   Formulas for spheres and cones will be provided.

4. (3.10) Linear approximations and differentials:

   $$\begin{array}{rll} L(x) & = f(a) + f'(a)(x-a) & \text{Linearization of } f \text{ at } a \\ \Delta y & = f(x+\Delta x) - f(x) & \text{Actual change of } f \\ dy & = f'(x)\,dx & \text{The differential - Used to approximate } \Delta y \end{array}$$

   The *relative* change in y is $\Delta y/y$ and was approximated by dy/y.

5. Main Theorems to know:

   - The Intermediate Value Theorem (this is from a while back, but we still use it).
   - The Extreme Value Theorem.

- The Mean Value Theorem.
- Rolle's Theorem.

6. Analyzing a function through its derivatives:

   (a) Use the derivatives and/or graph of the derivatives to determine where f is inc, dec, CU, CD.

   (b) Define Critical Points (CPs) and Inflection Points.

   (c) Understand the relationship between CPs and local extrema (Fermat's Theorem).

   (d) The first derivative test.

   (e) The second derivative test.

   (f) The closed interval method for finding absolute extrema.

7. (Section 4.8) Newton's Method for finding solutions to $f(x) = 0$:

   Begin with an initial guess x_0. We then perform the following to compute a sequence $x_1, x_2, \cdots$ that we hope will converge to the actual root:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

   This is actually computing the $x-$intercept of the tangent line thru $(x_n, f(x_n))$.

Review Questions

This is not meant to be an all exhaustive list, rather it is meant to give you some random questions out of context. It is important that you understand the homework and the quizzes as well as these exercises.

1. Suppose that a certain bacterial population grows to three times the initial population in 30 days.

   (a) What is the doubling time?

   (b) If the initial population is 10, how long does it take the population to reach 1000?

2. We can use "marginal revenue" just as we used marginal cost. If $R(x)$ is the revenue for producing x items, then

$$R'(x) = \lim_{h\to 0} \frac{R(x+h) - R(x)}{h} \approx R(x+1) - R(x)$$

   Therefore, the marginal revenue (the derivative of revenue) at x is an approximation to the revenue you get by producing one more unit.

   Suppose we have a toy company that can produce x toys per week, where $0 \le x \le 800$. The cost to produce the toys is a fixed cost of \$100 per week, and the cost per toy is \$4.50. The weekly revenue is $R(x) = 10x - 0.01x^2$. Profit is found by taking revenue and subtracting cost.

   (a) Find the marginal cost and revenue.

   (b) Compare $R'(100)$ to the revenue we get for the 101st item (per week).

   (c) Find the profit function, and find its maximum (note the domain) using the Closed Interval Method.

3. Short Answer:

   (a) Give the definition of a **critical point** for a function f:

   (b) State the three "Value Theorems" (don't just name them, but also state each):

   (c) What is the procedure for finding the maximum or minimum of a function $y = f(x)$ on a closed interval, $[a, b]$.

   (d) How do we determine if a function has a local maximum or minimum?

   (e) What is meant by *linearizing* a function?

4. True or False, and give a short reason:

   (a) If $f'(a) = 0$, then there is a local maximum or local minimum at $x = a$.

   (b) There is a vertical asymptote at $x = 2$ for $\frac{\sqrt{x^2+5}-3}{x^2-2x}$

   (c) If f has a global minimum at $x = a$, then $f'(a) = 0$.

   In the following, "increasing" or "decreasing" will mean for all real numbers x:

(d) If $f(x)$ is increasing, and $g(x)$ is increasing, then $f(x) + g(x)$ is increasing.

(e) If $f(x)$ is increasing, and $g(x)$ is increasing, then $f(x)g(x)$ is increasing.

(f) If $f(x)$ is increasing, and $g(x)$ is decreasing, then $f(g(x))$ is decreasing.

5. Find the global maximum and minimum of the given function on the interval provided:

   (a) $f(x) = \sqrt{9 - x^2}$, $[-1, 2]$

   (b) $g(x) = x - 2\cos(x)$, $[-\pi, \pi]$

6. Find the regions where f is increasing/decreasing: $f(x) = \frac{x}{(1+x)^2}$

7. For each function below, determine (i) where f is increasing/decreasing, (ii) where f is concave up/concave down, and (iii) find the local extrema.

   (a) $f(x) = x^3 - 12x + 2$

   (b) $f(x) = x\sqrt{6 - x}$

   (c) $f(x) = x - \sin(x)$, $0 < x < 4\pi$

8. Suppose $f(3) = 2$, $f'(3) = \frac{1}{2}$, and $f'(x) > 0$ and $f''(x) < 0$ for all x.

   (a) Sketch a possible graph for f.

   (b) How many roots does f have? (Explain):

   (c) Is it possible that $f'(2) = 1/3$? Why?

9. Let $f(x) = 2x + e^x$.

   (a) Show that f has exactly one real root.

   (b) Use Newton's Method three times to approximate the root.

10. Compute Δy and dy for the given x and $dx = \Delta x$. Sketch a diagram and label Δx, Δy and dy, if $f(x) = 6 - x^2$, $x = -2$, $\Delta x = 1$.

11. Linearize at $x = 0$:
$$y = \sqrt{x+1}e^{-x^2}$$
    Use the linearization to estimate $\sqrt{\frac{3}{2}}e^{-\frac{1}{4}}$

12. Let $f(x) = \sqrt{x} - \frac{x}{3}$ on $[0, 9]$. Verify that the function satisfies all the hypotheses of Rolle's Theorem, then find the values of c that satisfy its conclusion.

13. Let $f(x) = x^3 - 3x + 2$ on the interval $[-2, 2]$. Verify that the function satisfies all the hypotheses of the Mean Value Theorem, then find the values of c that satisfy its conclusion.

14. Let $f(x) = \tan(x)$. Show that $f(0) = f(\pi)$, but there is no number c in $(0, \pi)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

15. A child is flying a kite. If the kite is 90 feet above the child's hand level and the wind is blowing it on a horizontal course at 5 feet per second, how fast is the child paying out cord when 150 feet of cord is out? (Assume that the cord forms a line- actually an unrealistic assumption).

16. At 1:00 PM, a truck driver picked up a fare card at the entrance of a tollway. At 2:15 PM, the trucker pulled up to a toll booth 100 miles down the road. After computing the trucker's fare, the toll booth operator summoned a highway patrol officer who issued a speeding ticket to the trucker. (The speed limit on the tollway is 65 MPH).

    (a) The trucker claimed that she hadn't been speeding. Is this possible? Explain.

    (b) The fine for speeding is \$35.00 plus \$2.00 for each mph by which the speed limit is exceeded. What is the trucker's minimum fine?

17. Let $f(x) = \frac{1}{x}$

    (a) What does the Extreme Value Theorem (EVT) say about f on the interval $[0.1, 1]$?

    (b) Although f is continuous on $[1, \infty)$, it has no minimum value on this interval. Why doesn't this contradict the EVT?

18. Let f be a function so that $f(0) = 0$ and $\frac{1}{2} \leq f'(x) \leq 1$ for all x. Explain why $f(2)$ cannot be 3 (Hint: You might use a value theorem to help).

19. Estimate the change in the indicated quantity using differentials.

    (a) The volume, $V = s^3$ of a cube, if its side length s is increased from 5 inches to 5.1 inches.

    (b) The volume, $V = \frac{4}{3}\pi r^3$ of a sphere, if its radius changes from 2 to 2.1

    (c) The volume, $V = \frac{1000}{p}$, of a gas, if the pressure p is decreased from 100 to 99.

    (d) The period of oscillation, $T = 2\pi\sqrt{\frac{L}{32}}$, of a pendulum, if its length L is increased from 2 to 2.2.

20. Related Rates Extra Practice:

    (a) The top of a 25-foot ladder, leaning against a vertical wall, is slipping down the wall at a rate of 1 foot per second. How fast is the bottom of the ladder slipping along the ground when the bottom of the ladder is 7 feet away from the base of the wall?

    (b) A 5-foot girl is walking toward a 20-foot lamppost at a rate of 6 feet per second. How fast is the tip of her shadow (cast by the lamppost) moving?

    (c) Under the same conditions as above, how fast is the length of the girl's shadow changing?

    (d) A rocket is shot vertically upward with an initial velocity of 400 feet per second. Its height s after t seconds is $s = 400t - 16t^2$. How fast is the distance changing from the rocket to an observer on the ground 1800 feet away from the launch site, when the rocket is still rising and is 2400 feet above the ground?

(e) A small funnel in the shape of a cone is being emptied of fluid at the rate of 12 cubic centimeters per second (the tip of the cone is downward). The height of the cone is 20 cm and the radius of the top is 4 cm. How fast is the fluid level dropping when the level stands 5 cm above the vertex of the cone [The volume of a cone is $V = \frac{1}{3}\pi r^2 h$].

(f) A balloon is being inflated by a pump at the rate of 2 cubic inches per second. How fast is the diameter changing when the radius is $\frac{1}{2}$ inch? ($V = \frac{4}{3}\pi r^3$)

(g) A rectangular trough is 8 feet long, 2 feet across the top, and 4 feet deep. If water flows in at a rate of 2 cubic feet per minute, how fast is the surface rising when the water is 1 foot deep?

(h) If a mothball (sphere) evaporates at a rate proportional to its surface area $4\pi r^2$, show that its radius decreases at a constant rate.

(i) If an object is moving along the curve $y = x^3$, at what point(s) is the $y-$coordinate changing 3 times more rapidly than the $x-$coordinate?

21. **Graphical Exercises**

- 4.8: 4, 5
- 4.3: 5-6, 7-8, 31-32
- 4.2: 7
- 4.1: 3-6
- 3.10: 19. 20
- 3.7: 5, 6