Overview of Calculus I

The Limit

1. Main definition: We say that $\lim_{x \to a} f(x) = L$ if for every $\epsilon > 0$, there is a $\delta > 0$ so that

$$0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \epsilon$$

2. Verbal description of the limit: We say that $\lim_{x \to a} f(x) = L$ if we can make the values of $f(x)$ arbitrarily close to the number L by taking x sufficiently close to the number a (excluding the point at $x = a$).

3. Definitions (this is about the notation): $\lim_{x \to a^+} f(x) = L, \quad \lim_{x \to a^-} f(x) = L$

4. Graphical Methods: Be able to compute a limit graphically. For the ϵ, δ definition, be able to do graphical exercises like 1-4 on p. 117 (Sect 2.4).

5. Algebraic Methods we discussed for computing a limit:
   (a) Simplify
   (b) Factor and Cancel
   (c) Multiply by Conjugate
   (d) Divide by x^n (Mainly for $x \to \infty$)
   (e) L'Hospital's Rule: For $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

   $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

   Other forms for l'Hospital (see p. 306-307, Sect 4.4)

   $$f(x)g(x) = \frac{f(x)}{1/g(x)} \qquad\qquad f(x)^{g(x)} = \mathrm{e}^{g(x)\ln(f(x))}$$

6. Horizontal/Vertical Asymptotes:
   (a) $x = a$ is a vertical asymptote for $f(x)$ if one of the following limits is infinite: $\lim_{x \to a^\pm} f(x)$
   (b) $y = b$ is a horizontal asymptote for $f(x)$ if one of the following is true: $\lim_{x \to \pm\infty} f(x) = b$

7. Heuristics that can be used:
   (a) "$\infty + \infty = \infty$", but $\infty - \infty$ is not necessarily 0 (it is indeterminant).
   (b) If the denominator goes to zero, but the numerator does not, the limit is $\pm\infty$.
   (c) If the denominator goes to $\pm\infty$, and the numerator does not, the overall limit goes to zero.
   (d) Other indeterminate forms we discussed:

   $$\frac{0}{0} \qquad \frac{\pm\infty}{\pm\infty} \qquad 0 \cdot \infty \qquad 1^\infty$$

8. Limit Laws (Sect 2.3)- You need not memorize them, but be able to apply them- For example, see the section of True/False questions.

Continuity

1. Definition:
$$\lim_{x\to a} f(x) = f(a)$$

2. Interpretation of the definition: It means 3 things must be true: (1) $f(a)$ exists, (2) The limit exists, and (3) Items 1 and 2 are the same number.

3. IVT: If f is continuous on $[a, b]$, and w is a number between $f(a)$ and $f(b)$, there is at least one c in $[a, b]$ so that $f(c) = w$.

   Interpretations:

   (a) If f is continuous, the range of a closed interval is an interval.

   (b) If f is continuous, and $f(x_1) > 0, f(x_2) < 0$, then there is a c between x_1 and x_2 where $f(c) = 0$ (f has at least one root in the interval between x_1 and x_2).

4. Continuous v. Differentiable: If f is differentiable at $x = a$, it is continuous at $x = a$. If f is continuous at $x = a$, we don't know if it is differentiable at $x = a$. That is, "All differentiable functions are continuous, but not all continuous functions are differentiable".

The Derivative

1. Definition:
$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

2. Be able to compute f' using the definition (and without resorting to l'Hospital's rule!).

3. Rules for Differentiation (See table)

   Highlighted techniques: Logarithmic differentiation, Implicit differentiation.

4. Higher derivatives: (a) Differential Equations, (b) Velocity and Acceleration, (c) Inc/Dec, concave up/down

5. Derivative of Inverses:

   (a) Be able to derive the derivative formula for the inverse trig functions (we needed implicit differentiation and triangles for this).

   (b) If the point (a, b) is on the graph of f, then: (1) the point (b, a) is on the graph of f^{-1}, and (2) $f'(b) = \frac{1}{f'(a)}$.

6. Equation of the Tangent Line: Please remember that $f'(x)$ gives a *FORMULA* for the slope, and is not the slope itself!!

Using the Derivative

Three Important Theorems

1. Rolle's Theorem: If f is continuous on $[a, b]$ and f is differentiable on (a, b), and $f(a) = f(b)$, then there is a c between a and b for which $f'(c) = 0$.

   Corollary: Rolle's Theorem gives us a connection between the roots of a function and its derivatives:

   - If $f'(x) \neq 0$ on (a, b), then $f(x)$ can have at most 1 root on $[a, b]$.
   - If $f'(x) = 0$ has one solution on (a, b), then f can have at most 2 roots on $[a, b]$.
   - If $f'(x) = 0$ has two solutions on (a, b), then f can have at most 3 roots on $[a, b]$.

and so on.

2. MVT: If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a c in the interval (a, b) so that
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$
Interpretations and Useage:
   (a) There is a tangent line with the same slope as the line between the points $(a, f(a))$ and $(b, f(b))$.
   (b) If a car has an average (mean) velocity of N mph, then at some point, the speedometer read exactly N.
3. The Racetrack Principle: If $f(a) = g(a)$, and $f'(x) > g'(x)$ for all x in $[a, b]$, then $f(x) > g(x)$ for x in $[a, b]$.

Approximations and the Differential

1. Linear Approximations: The **linearization** of f at $x = a$ is the tangent line approximation to f at $x = a$: $y - f(a) = f'(a)(x - a)$.
2. Differentials: If x changes from x to Δx, then there is a corresponding change in f:
$$\Delta y = f(x + \Delta x) - f(x)$$
This is approximated by the differential $dy = f'(x)\,dx$.
Be able to label these quantities (Like Section 3.10, Exercises 19-22).

Optimization

1. EVT: If f is continuous on $[a, b]$, then f will attain a global max and global min on $[a, b]$. These points will be either at critical points or at endpoints.
2. Local: Let $x = a$ be a critical point of f.
   (a) First Derivative Test:
      i. If $f'(x)$ changes sign at $x = a$, then we have a local min (if changes from $-$ to $+$), or a local max (if changes from $+$ to $-$).
      ii. If $f'(x)$ does not change sign at $x = a$, we have neither.
   (b) Second Derivative Test: Let $f'(a) = 0$. Then
      i. If $f''(a) > 0$, we have a local min at $x = a$.
      ii. If $f''(a) < 0$, we have a local max at $x = a$.
      iii. If $f''(a) = 0$, the test is inconclusive.
3. Global:
   (a) On a closed interval: Use the EVT, build a chart using endpoints and critical points.
   (b) Not a closed interval: If $f'(x)$ only changes sign once at $x = a$, then $x = a, y = f(a)$ is either a global min ($-$ to $+$) or a global max ($+$ to $-$).

Related Rates

Main idea here was to think of all variables as depending on time. Relate the derivative of one quantity to the derivative of the other quantity. Similar triangles and the Pythagorean Theorem played a big role here.

Modeling

We looked at exponential growth/decay models (you should know that $y' = ry$ leads to $y(t) = P_0 e^{rt}$). This model also is used for interest, when it is continuously compounding. We also looked at the derivative in physics and economics, in particular we considered the marginal cost.

Finally, we constructed mathematical models when setting up the applied max and min problems (like minimizing cost to construct a box, or minimize time to travel from a boat to a lighthouse, etc.).

Newton's Method

Used to solve $f(x) = 0$ for x. You begin with an initial guess: x_0, and refine this estimate:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Graphically, these values are the $x-$intercepts of the tangent lines to f at $(x_i, f(x_i))$.

Antiderivatives

Know the definition of an antiderivative F. Know the table of antiderivatives. Be able to solve "simple" differential equations (like # 25-48 in Sect 4.9). Given the graph of f', be able to sketch f. Given three graphs, determine which is f, f', and f''.

Tables

- Derivative formulas:

f	f'
cf	cf'
$f \pm g$	$f' \pm g'$
fg	$f'g + fg'$
$f(g(x))$	$f'(g(x))g'(x)$
$\frac{f}{g}$	$\frac{f'g - fg'}{g^2}$
$f(x)^{g(x)}$	Use log diff.

f	f'
c	0
x^n	nx^{n-1}
e^x	e^x
a^x	$a^x \ln(a)$
$\ln\|x\|$	$\frac{1}{x}$
$\log_a(x)$	$\frac{1}{x} \cdot \frac{1}{\ln(a)}$

f	f'
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\cot(x)$	$-\csc^2(x)$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$

- Antiderivative Formulas:

f	F
cf	cF
$f \pm g$	$F \pm G$

f	F
c	cx
x^n	$\frac{1}{n+1}x^{n+1}$
$1/x$	$\ln\|x\|$
e^x	e^x

f	F
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$
$\sec^2(x)$	$\tan(x)$
$\sec(x)\tan(x)$	$\sec(x)$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)$
$\frac{1}{1+x^2}$	$\tan^{-1}(x)$