Extra Practice: Optimization (4.7)

The following are some extra practice problems for "optimization", section 4.7. Solutions to be posted soon.

1. Suppose we have two numbers, one is a positive number and the other is its reciprocal. Find the two numbers so that the sum is small as possible.

   SOLUTION: Let x and $1/x$ be the two numbers. Then we want to find the minimum of:
   $$f(x) = x + \frac{1}{x} \qquad x \geq 0$$
   If we look at the critical points of f, we see that $x = \pm 1$ (but we only take $x = 1$). Further, the derivative is:
   $$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$
   We see that $f'(x) < 0$ if $0 < x < 1$ and $f'(x) > 0$ if $x > 1$. By the first derivative test, $x = 1$ is the minimum.

2. Find two positive numbers such that their product is 16 and the sum is as small as possible.

   SOLUTION: Let x, y be the two numbers. We want to find the minimum of $x + y$, if $xy = 16$.

   Turning this into a function of one variable using substitution: $y = 16/x$, we have:
   $$f(x) = x + \frac{16}{x} \quad \Rightarrow \quad f'(x) = 1 - \frac{16}{x^2}$$
   Therefore, $x = 4$ is the critical points, and by the first derivative test, we see that $f'(x) < 0$ if $0 < x < 4$ and $f'(x) > 0$ if $x > 4$, so the minimum value is $4 + \frac{16}{4} = 4 + 4 = 8$.

3. A 20-inch piece of wire is bent into an L-shape. Where should the bend be made to minimize the distance between the two ends?

   SOLUTION: Think of the wire as being bent so the corner is at the origin. Then the two ends of the wire are at $(x, 0)$ and $(0, y)$, and we can interpret the problem as saying that we want to find the minimum of $x^2 + y^2$ (the square of the distance) such that $x + y = 20$. Using substitution,
   $$f(x) = x^2 + (20 - x)^2 \quad \Rightarrow \quad f'(x) = 2x - 2(20 - x) = 0 \quad \Rightarrow \quad 2x - 20 = 0 \quad \Rightarrow \quad x = 10$$
   Now, we should also note that $0 \leq x \leq 20$, so that at the endpoints, $f(0) = 20^2$ and $f(20) = 20^2$. We have $f(10) = 10^2$ which gives us our (global) minimum. Therefore, L-shape should be a square, with each end being 10 inches.

4. Find the point on the line $y = x$ closest to the point $(1, 0)$.

   SOLUTION: We recall that it is a bit easier to minimize the *square* of the distance, so we'll find the minimum of

   $$(x-1)^2 + y^2 \text{ such that } y = x$$

   Therefore, we want the minimum of

   $$f(x) = (x-1)^2 + x^2 = 2x^2 - 2x + 1 \quad \Rightarrow \quad f'(x) = 4x - 2 = 0 \quad \Rightarrow \quad x = 2$$

   We also note that if $x < 2$, then $f'(x) < 0$ and if $x > 2$, then $f'(x) > 0$. Therefore, we have found the minimum at $x = 2$. The point on the line $y = x$ that is closest to $(1, 0)$ is the point $(2, 2)$.

5. A box is constructed out of two different types of metal. The metal for the top and bottom, which are both square, costs \$1.00 per square foot, and the metal for the sides costs \$2.00 per square foot. Find the dimensions that minimize the cost of the box is the box must have a volume of 20 cubic feet.

   SOLUTION: Since the top/bottom are square, let those dimensions be x, x and the height be h. The cost of the box is determined by the surface area- The top and bottom each have area x^2, and each side has area xh. Therefore,

   $$\text{Cost } = 2x^2 \times \$1.00 + 4xh \times \$2.00 = 2x^2 + 8xh \text{ such that } V = x^2 h = 20$$

   Now we see that we can write the cost function as a function of x alone (using substitution)

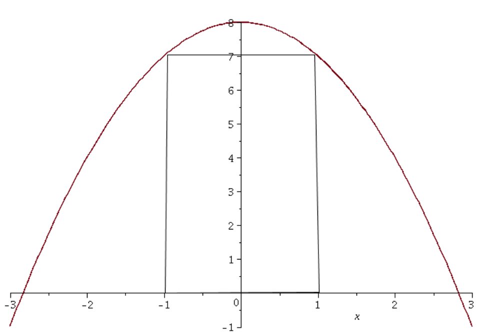
   $$C(x) = 2x^2 + 8x\frac{20}{x^2} = 2x^2 + \frac{160}{x} \quad \Rightarrow \quad C'(x) = 4x - \frac{160}{x^2} = 0$$

   Solving for x, we get

   $$4x^3 = 160 \quad \Rightarrow \quad x^3 = 40 \quad \Rightarrow \quad x = \sqrt[3]{40}$$

   And the height would be $160/(40^{2/3})$. We can check the first derivative as we did in the previous problems, C' changes sign from negative to positive at the critical point.

6. A rectangle is to be inscribed between the $x-$axis and the upper part of the graph of $y = 8 - x^2$ (symmetric about the $y-$axis). For example, one such rectangle might have vertices: $(1, 0), (1, 7), (-1, 7), (-1, 0)$ which would have an area of 14. Find the dimensions of the rectangle that will give the largest area.



   SOLUTION: Try drawing a picture first: The parabola opens down, goes through the $y-$intercept at 8, and has $x-$intercepts of $\pm\sqrt{8}$. Now, let x be as usual, so that the full length of the base of the rectangle is $2x$.

Then the height is y, or $8 - x^2$. Therefore, the area of the rectangle is:

$$A = 2xy = 2x(8 - x^2) = 16x - 2x^3$$

and $0 \le x \le \sqrt{8}$. We see that the area will be zero at the endpoints, so we expect a maximum at the critical point inside the interval:

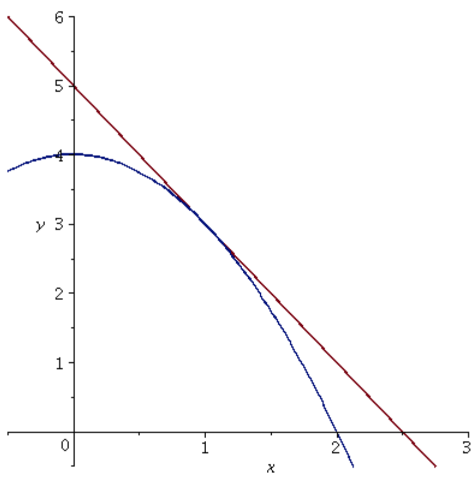
$$\frac{dA}{dx} = 16 - 6x^2$$

so the critical points are $x = \pm\sqrt{8/3}$, of which only the positive one is in our interval. So the dimensions of the rectangle are as follows (which give the maximum area of approx. 17.4):

$$2x = 2\sqrt{83} \qquad y = 8 - \frac{8}{3} = \frac{16}{3}$$

7. What is the smallest possible area of the triangle that is cut off by the first quadrant and whose hypotenuse is tangent to the curve $y = 4 - x^2$ at some point?

SOLUTION: See the figure below.



The area of the triangle is $A = \frac{1}{2}bh$, where the base is the length of the $x-$ intercept of the tangent line and the height is the $y-$intercept of the tangent line.
If the $x-$coordinate for the tangent line is given by $(a, f(a))$ and the slope is $f'(a)$, then the equation of the line is:

$$y - f(a) = f'(a)(x - a)$$

Further, we have $f(a) = 4 - a^2$ and $f'(a) = -2a$. To find the $x-$intercept, set $y = 0$ and solve for x.

For the $x-$intercept, we get

$$0 - (4 - a^2) = -2a(x - a) \quad \Rightarrow \quad x = \frac{a^2 + 4}{2a}$$

For the $y-$intercept, we get

$$y - (4 - a^2) = -2a(0 - a) \quad \Rightarrow \quad y = a^2 + 4$$

Therefore, the area is given by

$$A = \frac{1}{2}\left(\frac{a^2 + 4}{2a}\right)\left(a^2 + 4\right) = \frac{(a^2 + 4)^2}{4a}$$

Set the derivative to zero:

$$\frac{dA}{da} = a^2 + 4 - \frac{(a^2+4)^2}{4a^2} = 0 \quad \Rightarrow \quad 1 = \frac{a^2+4}{4a^2} \quad \Rightarrow \quad 4a^2 = a^2 + 4$$

From which $a = 2/\sqrt{3}$. If we look at $a > 0$, this is the only critical point, with the derivative changing sign from negative to positive (so we have a global minimum). Therefore, the smallest possible area is the following (on the exam, you could leave this unevaluated to save time).

$$A = \frac{((4/3)+4)^2}{4 \cdot 2/\sqrt{3}}$$

8. You're standing with Elvis (the dog) on a straight shoreline, and you throw the stick in the water. Let us label as "A" the point on the shore closest to the stick, and suppose that distance is 7 meters. Suppose that the distance from you to the point A is 10 meters. Suppose that Elvis can run at 3 meters per second, and can swim at 2 meters per second. How far along the shore should Elvis run before going in to swim to the stick, if he wants to minimize the time it takes him to get to the stick?

   SOLUTION: Recall that given distance d, rate r and time t, then $d = rt$, or $t = d/r$. If we let x be distance from point A that Elvis should travel before entering the water, then $10 - x$ is the distance that Elvis runs on the beach, and $\sqrt{x^2+49}$ is the distance Elvis has to swim. That gives the total time:

$$T(x) = \frac{\sqrt{x^2+49}}{2} + \frac{10-x}{3} \qquad 0 \le x \le 10$$

   We want to minimize T on the closed interval given, so first we find the critical points, then build a table. The critical points are:

$$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2+49}} - \frac{1}{3} = 0 \quad \Rightarrow \quad 3x = 2\sqrt{x^2+49} \quad \Rightarrow \quad x = \frac{14}{\sqrt{5}}$$

   (we discard the negative solution). Now build a table:

x	0	$14/\sqrt{5}$	10
T	6.833	5.94	6.10

   Therefore, Elvis minimizes his overall time by first running $10 - 14/\sqrt{5} \approx 3.74$ meters, then swimming the remaining 9.39 meters for a total of 6.10 seconds.

   NOTE: Since you wouldn't have a calculator on the exam, I would try to make sure the numbers worked out better than this.