Intro to Limits

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Overall, $\lim_{x \to 1} f(x) \approx 1.188$.

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Overall, $\lim_{x \to 1} f(x)$ does not exist.

Key Definition: The Limit

Suppose $f(x)$ is defined for all x near $x = a$, except perhaps at $x = a$. Then, if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a, but not equal to a, then we say that

$$\lim_{x \to a} f(x) = L$$

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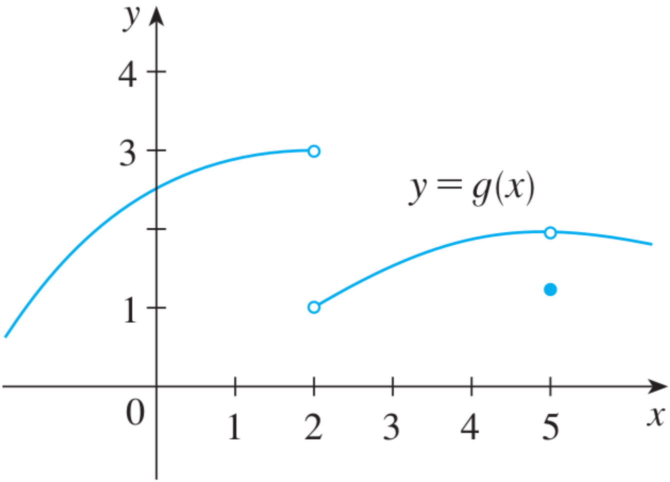
- Graphical limit. Using the graph of f, determine if the limit exists.
- Algebraic limit (most of the time)

Graphical Determination



Template Example

For $y = \dfrac{1}{x}$, find the limit as $x \to 0$, and as x gets really large.

- $1/(1/10) = 10$

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$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

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- $1/10$ and $1/100$ and $1/1000$, etc:
  "As $x \to \infty$, the $y-$values go to zero"

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

A Simple Example: 0/0

- Find $\lim_{x \to 0} \dfrac{x^2}{x}$
  SOLUTION: Since $x^2/x = x$, then we can simplify first:

$$\lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} x = 0$$

- Find $\lim_{x \to 0} \dfrac{x}{x^2}$
  SOLUTION: Since $x^2/x = x$, then we can simplify first:

$$\lim_{x \to 0} \frac{x}{x^2} = \lim_{x \to 0} \frac{1}{x} \quad \Rightarrow \quad DNE$$

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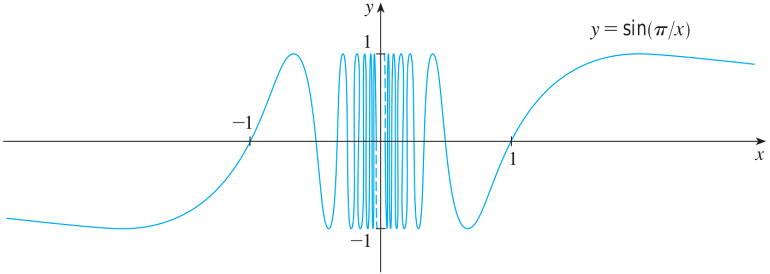
- If you have a fraction, and both numerator and denominator go to zero, we cannot conclude anything.
- If you have a fraction, and the numerator goes to a constant, but the denominator goes to infinity, then overall the fraction goes to zero.

An Odd Case

Consider $y = \sin(\pi/x)$ (Graph below). Does the limit exist as $x \to 0$?



Definition: Vertical Asymptote

The line $x = a$ is said to be a vertical asymptote for $f(x)$ if, as x approaches a from right or the left, the $y-$values go to $\pm\infty$:

$$\lim_{x \to a^{+/-}} f(x) = \pm\infty$$

(Only one of these needs to be true).

Sketch

Sketch an example of the graph of a function f that would satisfy (all) the following conditions:

$$\lim_{x\to 0} f(x) = 1 \qquad \lim_{x\to 3^-} f(x) = -2 \qquad \lim_{x\to 3^+} f(x) = 2$$

$$f(0) = -1 \qquad f(3) = 1$$

Algebraically Determining Limits- Limit Laws

The limit laws state that:

- The limit of a sum (or difference) is the sum (or difference) of the limits.
- The limit of a product is the product of the limits.
- The limit of a quotient is the quotient of the limits (provided the denominator does not go to zero!).

By these laws, we may conclude that, if $p(x)$ is any polynomial or rational function whose domain includes $x = a$, then

$$\lim_{x \to a} p(x) = p(a)$$

Examples

$$\lim_{x \to 3} \frac{x^2 - 1}{x + 1} =$$

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$$\lim_{x \to 3} \frac{x^2 - 1}{x + 1} = \frac{3^2 - 1}{3 + 1} = \frac{8}{4} = 2$$

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Examples

$$\lim_{x\to 3}\frac{x^2-1}{x+1}=\frac{3^2-1}{3+1}=\frac{8}{4}=2$$

$$\lim_{x\to -1}\frac{x^2-1}{x+1}=\lim_{x\to -1}\frac{(x+1)(x-1)}{(x+1)}=\lim_{x\to -1}(x-1)=-2$$

Algebraic Technique: Simplify and Cancel if possible!

(Example 5, 2.3) Find $\lim_{h\to 0} \frac{(3+h)^2-9}{h}$

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Algebraic Technique: "Rationalize"

(Example 5, 2.3) Find $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

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$$\lim_{t\to 0} \frac{(t^2+9)-9}{t^2(\sqrt{t^2+9}+3)} = \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3} = \frac{1}{6}$$