

SAMPLE EXAM 1 SOLUTIONS and COMMENTS

1. Give the definition: $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

Grading note: The limit is a *crucial* part of this definition. Also, remember to keep the arguments consistent- For example, if you're defining $f'(x)$, don't use $f(a+h)$ in the difference quotient.

2. State the Fundamental Theorem of Calculus. Let f be continuous on $[a, b]$.

- If $g(x) = \int_a^x f(t) dt$, then $g'(x) = f(x)$.
Note that this says that g is a specific antiderivative.
- Let F be any antiderivative of f . Then $\int_a^b f(x) dx = F(b) - F(a)$

3. True or False (and give a short reason):

- (a) If f is continuous at $x = a$, then f is differentiable at $x = a$.

SOLUTION: False. For example, $f(x) = |x|$ is continuous at $x = 0$, but the function is not differentiable at $x = 0$.

- (b) If $3 \leq f(x) \leq 5$ then $6 \leq \int_1^3 f(x) dx \leq 10$

SOLUTION: True, since

$$3 \leq f(x) \leq 5 \quad \Rightarrow \quad 3 \cdot (3 - 1) \leq \int_1^3 f(x) dx \leq 5 \cdot (3 - 1)$$

- (c) All continuous functions have antiderivatives.

SOLUTION: True. If $f(x)$ is continuous, then $g(x) = \int_a^x f(t) dt$ is an antiderivative by the FTC.

- (d) $\int_{-1}^2 -x^{-2} dx = \cdot$

SOLUTION: False. Since $-1/x^2$ is NOT continuous on $[-1, 2]$, we cannot use the FTC to evaluate the integral.

4. Find $f'(x)$ directly from the definition of the derivative (using limits and without using l'Hospital's rule):

$$f(x) = \sqrt{1+x}$$

SOLUTION: Once we set up the difference quotient, multiply by the conjugate:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\sqrt{1+x+h} - \sqrt{1+x}}{h} &\cdot \frac{\sqrt{1+x+h} + \sqrt{1+x}}{\sqrt{1+x+h} + \sqrt{1+x}} = \\ \lim_{h \rightarrow 0} \frac{(1+x+h) - (1+x)}{h(\sqrt{1+x+h} + \sqrt{1+x})} &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{1+x+h} + \sqrt{1+x})} = \frac{1}{2\sqrt{1+x}} \end{aligned}$$

5. Derive the formula for the derivative of $y = \sin^{-1}(x)$.

SOLUTION: First we re-write the function so we can implicitly differentiate it.

$$\sin(y) = x$$

This corresponds to a right triangle with an angle y , opposite length x , hypotenuse 1, and adjacent side $\sqrt{1-x^2}$ (by the Pythagorean Theorem). Now differentiate and convert back to x :

$$\cos(y)y' = 1 \quad \Rightarrow \quad y' = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$$

6. Find dy/dx (solve for it, if necessary):

(a) $y = \sin^3(x^2 + 1) + \tan^{-1}(x)$

SOLUTION: Note the triple composition, so we'll use the chain rule:

$$\frac{dy}{dx} = 3 \sin^2(x^2 + 1) \cos(x^2 + 1) \cdot 2x + \frac{1}{x^2 + 1}$$

(b) $y = 3^{1/x} + \sec(x)$

SOLUTION: $y' = 3^{1/x} \ln(3) \cdot (-x^{-2}) + \sec(x) \tan(x)$

(c) $\sqrt{x+y} = 4xy$

SOLUTION: Remember to use the product rule for the right side of the equation-

$$\frac{1+y'}{2\sqrt{x+y}} = 4y + 4xy'$$

Solving for y' , break up the fraction on the left and group y' terms together:

$$y' \left(\frac{1}{2\sqrt{x+y}} - 4x \right) = -\frac{1}{2\sqrt{x+y}} + 4y$$

Let's make these single fractions to make the solution clearer:

$$y' \left(\frac{1 - 8x\sqrt{x+y}}{2\sqrt{x+y}} \right) = \frac{-1 + 8y\sqrt{x+y}}{2\sqrt{x+y}} \Rightarrow y' = \frac{-1 + 8y\sqrt{x+y}}{1 - 8x\sqrt{x+y}}$$

Grading note: You should end up with a single fraction, and not a compound fraction. You may use terms like $(x+y)^{-1/2}$ in your fraction, although it would be nice to clean up the terms.

7. Find the limit, if it exists (you may use any technique from class):

(a) $\lim_{x \rightarrow 0} \frac{1 - e^{-2x}}{\sec(x)} = \frac{0}{1} = 0$

(b) $\lim_{x \rightarrow 4^+} \frac{x-4}{|x-4|} = 1$ NOTE: You should rewrite first as $\frac{x-4}{|x-4|} = \begin{cases} 1 & \text{if } x > 4 \\ -1 & \text{if } x < 4 \end{cases}$

(c) $\lim_{x \rightarrow -\infty} \sqrt{\frac{2x^2 - 1}{x + 8x^2}}$

SOLUTION: Divide numerator and denominator by x , and since $x < 0$, we will use the substitution $x = -\sqrt{x^2}$ to simplify. That is:

$$\lim_{x \rightarrow -\infty} \sqrt{\frac{2x^2 - 1}{x + 8x^2}} = \lim_{x \rightarrow -\infty} \frac{\sqrt{2x^2 - 1}/x}{\sqrt{x + 8x^2}/x} = \lim_{x \rightarrow -\infty} \frac{-\sqrt{\frac{2x^2 - 1}{x^2}}}{-\sqrt{\frac{x + 8x^2}{x^2}}} = \lim_{x \rightarrow -\infty} \frac{\sqrt{2 - \frac{1}{x^2}}}{\sqrt{\frac{1}{x} + 8}} = \frac{1}{2}$$

GRADING notes: Even though in this case it turned out that the negative signs canceled, that doesn't always happen. Also, don't leave your answer as $\sqrt{2/8}$ - Go ahead and simplify that to $1/2$.

(d) $\lim_{x \rightarrow 0^+} x^x$

For this, we'll need l'Hospital's rule. We need to manipulate the expression first.

$$x^x = e^{\ln(x^x)} = e^{x \ln(x)}$$

Therefore, we'll take the limit of the exponent first:

$$\lim_{x \rightarrow 0^+} x \ln(x) = \lim_{x \rightarrow 0^+} \frac{\ln(x)}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} -x = 0$$

Overall, the limit is e^0 , or 1.

8. Evaluate the Riemann sum by first writing it as an appropriate definite integral: $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \sqrt{1 + \frac{3i}{n}}$.

We see that a natural choice for $(b-a)/n$ is $3/n$, and $a + i\frac{b-a}{n} = 1 + \frac{3i}{n}$. In that case, $a = 1, b = 1 + 3 = 4$, and $f(x) = \sqrt{x}$. In that case,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \sqrt{1 + \frac{3i}{n}} = \int_1^4 \sqrt{x} dx = \left. \frac{2}{3} x^{3/2} \right|_1^4 = \frac{2}{3} (2^3 - 1^{3/2}) = \frac{14}{3}$$

Notice that there are an infinite number of equivalent choices here. For example,

$$\int_0^3 \sqrt{1+x} dx, \quad \int_2^5 \sqrt{x-1} dx, \quad \int_3^6 \sqrt{x-2} dx, \quad \text{etc.}$$

9. Differentiate: $F(x) = \int_{\sqrt{x}}^{x^2} \frac{t}{1+t} dt$

$$F'(x) = \frac{x^2}{1+x^2}(2x) - \frac{\sqrt{x}}{1+\sqrt{x}} \frac{1}{2\sqrt{x}}$$

This comes from the generalization to the FTC. If $g(x) = \int_a^x f(t) dt$, then

$$F(x) = \int_{h_1(x)}^{h_2(x)} f(t) dt = g(h_1(x)) - g(h_2(x)) \Rightarrow$$

$$F'(x) = g'(h_2(x))h_2'(x) - g'(h_1(x))h_1'(x) = f(h_2(x))h_2'(x) - f(h_1(x))h_1'(x)$$

10. Use the definition to evaluate $\int_0^3 1 + 3x dx$

$$\begin{aligned} \int_0^3 1 + 3x dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 + 3 \left(\frac{3i}{n} \right) \right) \frac{3}{n} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} + \frac{27i}{n^2} = \lim_{n \rightarrow \infty} \left(\frac{3}{n} \sum_{i=1}^n 1 + \frac{27}{n^2} \sum_{i=1}^n i \right) = \\ &= \lim_{n \rightarrow \infty} \left(\frac{3}{n} \cdot n + \frac{27}{n^2} \frac{n(n+1)}{2} \right) = 3 + \frac{27}{2} = \frac{33}{2} \end{aligned}$$

As a check, we can use FTC: $\int_0^3 1 + 3x dx = (x + (3/2)x^2)|_0^3 = 3 + \frac{27}{2} = \frac{33}{2}$

11. Use an appropriate substitution to evaluate $\int x^2 e^{x^3} dx$

SOLUTION: Let $u = x^3$, so $du = 3x^2 dx$, or $x^2 dx = \frac{1}{3} du$. Then

$$\int x^2 e^{x^3} dx = \frac{1}{3} \int e^u du = \frac{1}{3} e^u + C = \frac{1}{3} e^{x^3} + C$$

12. Evaluate, or find the general indefinite integral.

(a) $\int \sqrt{x^3} + \frac{1}{x^2+1} dx = \frac{2}{5} x^{5/2} + \tan^{-1}(x) + C$

(b) $\int_{-1}^1 t(1-t) dt = \left. \frac{1}{2} t^2 - \frac{1}{3} t^3 \right|_{-1}^1 = \left(\frac{1}{2} - \frac{1}{3} \right) - \left(\frac{1}{2} + \frac{1}{3} \right) = -\frac{2}{3}$

(c) $\int_0^1 5x - 5^x dx = \left. \frac{5}{2} x^2 - \frac{5^x}{\ln(5)} \right|_0^1 = \left(\frac{5}{2} - \frac{5}{\ln(5)} \right) - \left(0 - \frac{1}{\ln(5)} \right) = \frac{5}{2} - \frac{4}{\ln(5)}$

13. Evaluate:

(a) $\int_0^1 \frac{d}{dx} \left(e^{\tan^{-1}(x)} \right) dx = e^{\tan^{-1}(1)} - e^{\tan^{-1}(0)}$

Explanation: This is of the form $\int_0^1 f'(x) dx = f(1) - f(0)$, where $f(x) = e^{\tan^{-1}(x)}$.

(b) $\frac{d}{dx} \int_0^1 e^{\tan^{-1}(x)} dx = 0$

Explanation: Recall that $\int_0^1 f(x) dx$ is a number, and the derivative of a constant is zero.

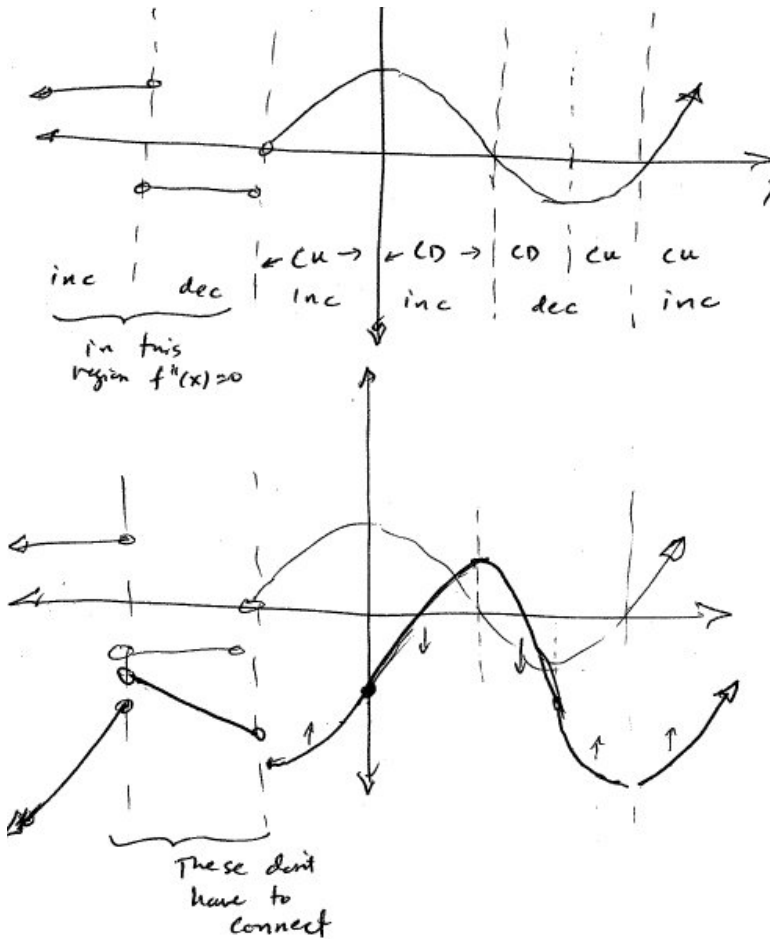
(c) $\frac{d}{dx} \int_0^x e^{\tan^{-1}(t)} dt = e^{\tan^{-1}(x)}$

Explanation: This is the FTC- If $g(x) = \int_0^x f(t) dt$, then $g'(x) = f(x)$.

14. Given the graph of the derivative, $f'(x)$, below, answer the following questions:

- (a) Find all intervals on which f is increasing.
- (b) Find all intervals on which f is concave up.
- (c) Sketch a possible graph of f if we require that $f(0) = -1$.

SOLUTION:



15. A rectangle is to be inscribed between the x -axis and the upper part of the graph of $y = 8 - x^2$ (symmetric about the y -axis). For example, one such rectangle might have vertices: $(1, 0)$, $(1, 7)$, $(-1, 7)$, $(-1, 0)$ which would have an area of 14. Find the dimensions of the rectangle that will give the largest area.

SOLUTION: Try drawing a picture first: The parabola opens down, goes through the y -intercept at 8, and has x -intercepts of $\pm\sqrt{8}$.

Now, let x be as usual, so that the full length of the base of the rectangle is $2x$. Then the height is y , or $8 - x^2$. Therefore, the area of the rectangle is:

$$A = 2xy = 2x(8 - x^2) = 16x - 2x^3$$

and $0 \leq x \leq \sqrt{8}$. We see that the area will be zero at the endpoints, so we expect a maximum at the critical point inside the interval:

$$\frac{dA}{dx} = 16 - 6x^2$$

so the critical points are $x = \pm\sqrt{8/3}$, of which only the positive one is in our interval. So the dimensions of the rectangle are as follows (which give the maximum area of approx. 17.4):

$$2x = 2\sqrt{8/3} \quad y = 8 - \frac{8}{3} = \frac{16}{3}$$

16. Find all values of c and d so that f is continuous at all real numbers:

$$f(x) = \begin{cases} 2x^2 - 1 & \text{if } x < 0 \\ cx + d & \text{if } 0 \leq x \leq 1 \\ \sqrt{x+3} & \text{if } x > 1 \end{cases}$$

Be sure it is clear from your work that you understand the definition of continuity.

SOLUTION: First, we note that f is continuous for all values of c, d if we remove $x = 0$ and $x = 1$ from the domain, so those are the only points we need to check. We'll check $x = 0$ first:

- $f(0)$ exists? Yes: $f(0) = 0 + d = d$ (Exists for all choices of c, d).
- Does the limit exist at $x = 0$? Check both directions:

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} 2x^2 - 1 = -1 \quad \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} cx + d = d$$

For the limit to exist overall, we must have: $d = -1$ (which will also make the limit equal to $f(0)$).

Check $x = 1$ now (and we'll go ahead and replace d with -1):

- $f(1)$ exists? Yes: $f(1) = c - 1$ (Exists for all choices of c, d).
- Does the limit exist at $x = 1$? Check both directions:

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} cx - 1 = c - 1 \quad \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \sqrt{x+3} = 2$$

For the limit to exist overall, we must have: $c - 1 = 2$, or $c = 3$

Therefore, if $c = 3$ and $d = -1$, f will be continuous at every x .