Exam 3: Applications of the Derivative

In this section of the course, we've looked at 3.9-3.10, then 4.1-4.4 and 4.7. Below is a short summary of the material.

1. Related Rates (3.9)

   The main things to recall here is how to use similar triangles. We looked at ways of converting the word problems into mathematical statements. Typical formulas might be area of a square, circle, or triangle. Anything beyond that and the formula would be given to you.

2. (3.10) Linear approximations and differentials.

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ \Delta y &= f(x+\Delta x) - f(x) \\ dy &= f'(x)\,dx \end{aligned}$$

   The first formula is the linearization of f. The second is the **actual** change in y as x changes from x to $x + \Delta x$. The third equation is the formula for the differential dy which is used to approximate Δy.

3. (4.1) Maximums and Minimums (Absolute or Global)

   In this section, we looked at how to find the absolute (or global) maximum and minimum values of a function on a closed interval.

   - The **Extreme Value Theorem** told us when to expect a global max/min.
   - To find the global max/min for f on a closed interval:
     - Compute the critical points for f.
     - Build a table using the critical points and endpoints for f.
     - The largest value in the table is the max, smallest is the min.

4. (4.2) The Mean Value Theorem

   In this section, we actually have two important theorems- Rolle's theorem and the Mean Value Theorem. You can remember Rolle's theorem by using the Mean Value Theorem and taking $f(a) = f(b) = 0$.

   Be sure you can state the MVT. Be able to "compute the c value from the MVT", Be able to show that an equation has at most n solutions. Given a bound on $f'(x)$, and $f(a)$, state how large or how small $f(b)$ is.

5. (4.3) The shapes of graphs (Local extrema)

   (a) Use the derivatives and/or graph of the derivatives to determine where f is inc, dec, CU, CD.
   (b) Define Critical Points (CPs) and Inflection Points.
   (c) Understand the relationship between CPs and local extrema (Fermat's Theorem).
   (d) The first derivative test.
   (e) The second derivative test.
   (f) The closed interval method for finding absolute extrema.

6. (4.4) l'Hospital's rule

   l'Hospital's rule gives us an extra technique for computing limits. The basic theorem concludes with:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

   Please know when this rule can be applied. We then looked at other forms that can will require some algebra before applying the rule. For example, the limit of a product $f(x)g(x)$ or the limit of an exponentiation, $f(x)^{g(x)}$.

7. (4.7) Optimization.

   I'm not sure how far we'll get into this section before the exam, but we should at least be able to set up the problem for questions up to 35 in the homework, and understand what techniques would be used to solve the problem.

Review Questions

This is not meant to be an all exhaustive list, rather it is meant to give you some random questions out of context. It is important that you understand the homework and the quizzes as well as these exercises. **Practice problems for 4.7 will be provided separately, once I know where we left off.**

1. Short Answer:
   (a) Give the definition of a **critical point** for a function f:
   (b) State the three "Value Theorems" (don't just name them, but also state each):
   (c) What is the procedure for finding the maximum or minimum of a function $y = f(x)$ on a closed interval, $[a, b]$.
   (d) How do we determine if a function has a local maximum or minimum?
   (e) What is meant by *linearizing* a function?

2. True or False, and give a short reason:
   (a) If $f'(a) = 0$, then there is a local maximum or local minimum at $x = a$.
   (b) There is a vertical asymptote at $x = 2$ for $\frac{\sqrt{x^2+5}-3}{x^2-2x}$
   (c) If f has a global minimum at $x = a$, then $f'(a) = 0$.
   (d) If $f''(2) = 0$, then $(2, f(2))$ is an inflection point for f.

   In the following, "increasing" or "decreasing" will mean for all real numbers x:
   (e) If $f(x)$ is increasing, and $g(x)$ is increasing, then $f(x) + g(x)$ is increasing.
   (f) If $f(x)$ is increasing, and $g(x)$ is increasing, then $f(x)g(x)$ is increasing.
   (g) If $f(x)$ is increasing, and $g(x)$ is decreasing, then $f(g(x))$ is decreasing.

3. Find the global maximum and minimum of the given function on the interval provided:
   (a) $f(x) = \sqrt{9 - x^2}$, $[-1, 2]$
   (b) $g(x) = x - 2\cos(x)$, $[-\pi, \pi]$

4. Find the regions where f is increasing/decreasing: $f(x) = \frac{x}{(1+x)^2}$

5. For each function below, determine (i) where f is increasing/decreasing, (ii) where f is concave up/concave down, and (iii) find the local extrema.
   (a) $f(x) = x^3 - 12x + 2$
   (b) $f(x) = x\sqrt{6 - x}$
   (c) $f(x) = x - \sin(x)$, $0 < x < 4\pi$

6. Suppose $f(3) = 2$, $f'(3) = \frac{1}{2}$, and $f'(x) > 0$ and $f''(x) < 0$ for all x.
   (a) Sketch a possible graph for f.

(b) How many roots does f have? (Explain):

(c) Is it possible that $f'(2) = 1/3$? Why?

7. Let $f(x) = 2x + \mathrm{e}^x$. Show that f has exactly one real root.

8. Suppose that $1 \le f'(x) \le 3$ for all $0 \le x \le 2$, and $f(0) = 1$. What is the largest and smallest that $f(2)$ can possibly be?

9. Linearize at $x = 0$: $y = \sqrt{x+1}\mathrm{e}^{-x^2}$. Use the linearization to estimate $\sqrt{\frac{3}{2}}\mathrm{e}^{-\frac{1}{4}}$.

10. Let $f(x) = \sqrt{x} - \frac{x}{3}$ on $[0, 9]$. Verify that the function satisfies all the hypotheses of the Mean Value Theorem, then find the values of c that satisfy its conclusion.

11. Let $f(x) = x^3 - 3x + 2$ on the interval $[-2, 2]$. Verify that the function satisfies all the hypotheses of the Mean Value Theorem, then find the values of c that satisfy its conclusion.

12. Let $f(x) = \tan(x)$. Show that $f(0) = f(\pi)$, but there is no number c in $(0, \pi)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

13. Find the limit, if it exists.

(a) $\lim_{x\to 0} \dfrac{\sin^{-1}(x)}{x}$

(b) $\lim_{x\to 0} \dfrac{x3^x}{3^x - 1}$

(c) $\lim_{x\to 0^+} \sin(x)\ln(x)$

(d) $\lim_{x\to 0} \cot(2x)\sin(6x)$

(e) $\lim_{x\to 0^+} x^{\sqrt{x}}$

(f) $\lim_{x\to 0^+} (4x+1)^{\cot(x)}$

14. Verify the given linear approximation (for small x).

(a) $\sqrt[4]{1+2x} \approx 1 + \frac{1}{2}x$

(b) $\mathrm{e}^x\cos(x) \approx 1 + x$

15. A child is flying a kite. If the kite is 90 feet above the child's hand level and the wind is blowing it on a horizontal course at 5 feet per second, how fast is the child paying out cord when 150 feet of cord is out? (Assume that the cord forms a line- actually an unrealistic assumption).

16. At 1:00 PM, a truck driver picked up a fare card at the entrance of a tollway. At 2:15 PM, the trucker pulled up to a toll booth 100 miles down the road. After computing the trucker's fare, the toll booth operator summoned a highway patrol officer who issued a speeding ticket to the trucker. (The speed limit on the tollway is 65 MPH).

(a) The trucker claimed that she hadn't been speeding. Is this possible? Explain.

(b) The fine for speeding is \$35.00 plus \$2.00 for each mph by which the speed limit is exceeded. What is the trucker's minimum fine?

17. Let $f(x) = \frac{1}{x}$

(a) What does the Extreme Value Theorem (EVT) say about f on the interval $[0.1, 1]$?

(b) Although f is continuous on $[1, \infty)$, it has no minimum value on this interval. Why doesn't this contradict the EVT?

18. Let f be a function so that $f(0) = 0$ and $\frac{1}{2} \le f'(x) \le 1$ for all x. Explain why $f(2)$ cannot be 3 (Hint: You might use a value theorem to help).

19. Related Rates Extra Practice:

(a) The top of a 25-foot ladder, leaning against a vertical wall, is slipping down the wall at a rate of 1 foot per second. How fast is the bottom of the ladder slipping along the ground when the bottom of the ladder is 7 feet away from the base of the wall?

(b) A 5-foot girl is walking toward a 20-foot lamppost at a rate of 6 feet per second. How fast is the tip of her shadow (cast by the lamppost) moving?

(c) Under the same conditions as above, how fast is the length of the girl's shadow changing?

(d) A rocket is shot vertically upward with an initial velocity of 400 feet per second. Its height s after t seconds is $s = 400t - 16t^2$. How fast is the distance changing from the rocket to an observer on the ground 1800 feet away from the launch site, when the rocket is still rising and is 2400 feet above the ground?

(e) A small funnel in the shape of a cone is being emptied of fluid at the rate of 12 cubic centimeters per second (the tip of the cone is downward). The height of the cone is 20 cm and the radius of the top is 4 cm. How fast is the fluid level dropping when the level stands 5 cm above the vertex of the cone [The volume of a cone is $V = \frac{1}{3}\pi r^2 h$].

(f) A balloon is being inflated by a pump at the rate of 2 cubic inches per second. How fast is the diameter changing when the radius is $\frac{1}{2}$ inch? ($V = \frac{4}{3}\pi r^3$)

(g) A rectangular trough is 8 feet long, 2 feet across the top, and 4 feet deep. If water flows in at a rate of 2 cubic feet per minute, how fast is the surface rising when the water is 1 foot deep?

(h) If a mothball (sphere) evaporates at a rate proportional to its surface area $4\pi r^2$, show that its radius decreases at a constant rate.

(i) If an object is moving along the curve $y = x^3$, at what point(s) is the $y-$coordinate changing 3 times more rapidly than the $x-$coordinate?

20. **Graphical Exercises** Please look these problems over as well- They include some graphical analysis.

- 4.3: 5-6, 7-8, 31-32
- 4.2: 7