Exam 3 Review solutions

1. Short Answer:

   (a) Give the definition of a **critical point** for a function f:
   A critical point is a value of x for which $f'(x) = 0$ or does not exist (from the domain of f).

   (b) State the three "Value Theorems" (don't just name them, but also state each):

   - Intermediate Value Theorem: If f is continuous on $[a, b]$ and N is any number between $f(a)$ and $f(b)$, then there is a c in $[a, b]$ such that $f(c) = N$.
   - Extreme Value Theorem: If f is continuous on $[a, b]$, then f attains an absolute maximum and minimum on $[a, b]$.
   - Mean Value Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a c in (a, b) such that

   $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

   (c) What is the procedure for finding the maximum or minimum of a function $y = f(x)$ on a closed interval, $[a, b]$.
   SOLUTION: This is the closed interval method. Find the critical points (in the interval $[a, b]$. Build a table using the CPs and endpoints. The largest $y-$value is your absolute max, the smallest $y-$value is your absolute min.

   (d) How do we determine if a function has a local maximum or minimum?
   SOLUTION: First we compute the critical points. Then we can use either the first or second derivative test. The first derivative test says that if the sign of the first derivative changes from positive to negative, we have a local maximum. If the sign changes from negative to positive, we have a local minimum. If there is no sign change, we do not have a local extreme point.
   For the second derivative test, if $f'(c) = 0$, we check $f''(c)$. If that value is positive, then c is where a local minimum occurs. If that value is negative, then c is where a local maximum occurs.

   (e) What is meant by *linearizing* a function?
   SOLUTION: We replace f by $L(x)$ close by some base point $x = a$. In this case, we say

   $$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

2. True or False, and give a short reason:

   (a) If $f'(a) = 0$, then there is a local maximum or local minimum at $x = a$.
   FALSE: For example, $f(x) = x^3$ at $x = 0$.

   (b) There is a vertical asymptote at $x = 2$ for $\frac{\sqrt{x^2+5}-3}{x^2-2x}$
   NOTE: I suspect it is FALSE because both the numerator and denominator are zero at $x = 2$. To show it, I need to take the limit:

   $$\lim_{x\to 2} frac\sqrt{x^2+5} - 3x^2 - 2xfrac\sqrt{x^2+5} + 3\sqrt{x^2+5} + 3 = \lim_{x\to 2} frac(x+2)(x-2)x(x-2)(\sqrt{x^2+5}+3) = \frac{1}{3}$$

   so, it is FALSE.

   (c) If f has a global minimum at $x = a$, then $f'(a) = 0$.
   FALSE. It could be that $x = a$ is an endpoint for an interval, or a point where $f'(a)$ does not exist.

   In the following, "increasing" or "decreasing" will mean for all real numbers x:

(d) If $f(x)$ is increasing, and $g(x)$ is increasing, then $f(x) + g(x)$ is increasing.
TRUE: Since f, g are increasing, f', g' are both positive so $f' + g'$ is positive as well.

(e) If $f(x)$ is increasing, and $g(x)$ is increasing, then $f(x)g(x)$ is increasing.
FALSE: It might be true, but

$$(fg)' = f'g + fg'$$

so we might have that $g < 0$ and $f < 0$, for example. In that case, the derivative would be negative.

(f) If $f(x)$ is increasing, and $g(x)$ is decreasing, then $f(g(x))$ is decreasing.
TRUE:

$$(f(g(x)))' = f'(g(x))g'(x)$$

so f' (evaluated at $g(x)$) is positive, and g' is positive.

3. Find the global maximum and minimum of the given function on the interval provided:

(a) $f(x) = \sqrt{9 - x^2}$, $[-1, 2]$
SOLUTION: $f'(x) = -x/\sqrt{9 - x^2}$, so we add the critical point $x = 0$. Build a table:

x	-1	0	2
y	2.82	3.0	2.23

So the global minimum is $y = 2.23$ and the global maximum is $y = 3.0$

(b) $g(x) = x - 2\cos(x)$, $[-\pi, \pi]$
SOLUTION: $g'(x) = 1 + 2\sin(x)$. Use a triangle and/or unit circle to find the values of $x = -5\pi/6$ and $x = -\pi/6$. Now the table:

x	$-\pi$	$-5\pi/6$	$-\pi/6$	π
y	-1.14	-0.88	-2.25	5.14

The global minimium is approx -2.25 and the global max is approx 5.14.

4. Find the regions where f is increasing/decreasing: $f(x) = \frac{x}{(1+x)^2}$

SOLUTION: Simplifying the derivative, we get:

$$f'(x) = \frac{1 - x}{(1 + x)^3}$$

Sign chart (include $x = -1$ although it is a vertical asymptote):

$f'(x)$	$-$	$+$	$-$
	$x < -1$	$-1 < x < 1$	$x > 1$

Therefore, f is decreasing if $x < -1$ and if $x > 1$, and increasing if $-1 < x < 1$.

5. For each function below, determine (i) where f is increasing/decreasing, (ii) where f is concave up/concave down, and (iii) find the local extrema.

(a) $f(x) = x^3 - 12x + 2$ (See Exercise 33, 4.3)
Hint: $f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x - 2)(x + 2)$, then look at a sign chart, and $f''(x) = 6x$.

(b) $f(x) = x\sqrt{6-x}$ (See Exercise 39, 4.3)
Hints: The domain is $x \leq 6$, and we can simplify

$$f'(x) = \frac{3}{2} \cdot \frac{x-4}{\sqrt{6-x}}$$

$$f''(x) = \frac{3}{4} \cdot \frac{x-8}{(6-x)^{3/2}}$$

(c) $f(x) = x - \sin(x)$, $0 < x < 4\pi$ (See Exercise 44. 4.3)
SOLUTION: $f'(x) = 1 - \cos(x)$, and $\cos(x) = 1$ at $x = 0, 2\pi, 4\pi$. Since we do not include $0, 4\pi$ in the interval, we have:

$f'(x)$	$+$	$+$
	$0 < x < 2\pi$	$2\pi < x < 4\pi$

Therefore, $f'(x) > 0$, and f is always increasing.
For concavity, $f''(x) = \sin(x)$. The sine is posiive (so f is increasing) on $(0, \pi)$ and $(2\pi, 3\pi)$. The sine is negative (so f is decreasing) on $(\pi, 2\pi)$ and $(3\pi, 4\pi)$.
For local extrema, the only critical point is 2π, and from our analysis of the first derivative, we see that this is a plateau, not a local extreme point (so there are no local extrema).

6. Suppose $f(3) = 2$, $f'(3) = \frac{1}{2}$, and $f'(x) > 0$ and $f''(x) < 0$ for all x.

   (a) Sketch a possible graph for f.
   SOLUTION: At the point $(3, 2)$, the function increases and is concave down.

   (b) How many roots does f have? (Explain):
   SOLUTION: The function f can have at most 1 root. If it had two roots, Rolle's theorem would mean that $f'(x) = 0$ for some x between the roots- But we're told that $f'(x) > 0$.

   (c) Is it possible that $f'(2) = 1/3$? Why?
   SOLUTION: Since $f'(3) = 1/2$ and $1/3 < 1/2$, then this implies that f' is increasing. However, that implies that $f'' > 0$, but $f''(x) < 0$ for all x. Therefore, the given value of $f'(x)$ is not possible.

7. Let $f(x) = 2x + e^x$.
   Show that f has exactly one real root.

   - We see that $f(-1) = -2 + e^{-1} < 0$ and $f(0) = 1 > 0$. Therefore, by the IVT there is at least one real root.
   - The derivative is $f'(x) = 2 + e^x$, which is always positive (since e^x is always greater than 0). Therefore, by Rolle's Theorem, there is at most 1 real root.
   - By the previous 2 items, there is exactly one (real) root.

8. Suppose that $1 \leq f'(x) \leq 3$ for all $0 \leq x \leq 2$, and $f(0) = 1$. What is the largest and smallest that $f(2)$ can be?
   SOLUTION: For the maximum,

$$\frac{f(2) - f(0)}{2 - 0} = f'(c) \leq 3 \quad \Rightarrow \quad \frac{f(2) - 1}{2} \leq 3 \quad \Rightarrow \quad f(2) \leq 6 + 1 = 7$$

   Similarly, for the minimum,

$$\frac{f(2) - f(0)}{2 - 0} = f'(c) \geq 1 \quad \Rightarrow \quad \frac{f(2) - 1}{2} \geq 1 \quad \Rightarrow \quad f(2) \leq 2 + 1 = 3$$

   Therefore, $3 \leq f(2) \leq 7$.

9. Linearize at $x = 0$:

$$y = \sqrt{x+1}\mathrm{e}^{-x^2}$$

Use the linearization to estimate $\sqrt{\frac{3}{2}}\mathrm{e}^{-\frac{1}{4}}$

SOLUTION: At $x = 0$, we have $y = \sqrt{1}\mathrm{e}^0 = 1$. Now for the slope:

$$f'(x) = \frac{1}{2}(x+1)^{-1/2}\mathrm{e}^{-x^2} + \sqrt{x+1}\mathrm{e}^{-x^2}(-2x)$$

so $f'(0) = \frac{1}{2}$, and $L(x) = 1 + \frac{1}{2}x$. Therefore, (note that the question asks you to approximate $f(1/2)$):

$$f(1/2) \approx L(1/2) = 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{4}$$

10. Let $f(x) = \sqrt{x} - \frac{x}{3}$ on $[0, 9]$. Verify that the function satisfies all the hypotheses of the MVT, then find the values of c that satisfy its conclusion.

SOLUTION: f is continuous on the given interval, and $f'(x)$ does not exist at 0, but that's OK since for Rolle's Theorem, we need f to be differentiable on $(0, 9)$. Finally, check that $f(0) = f(9) = 0$. To find the value of c, Rolle's Theorem guarantees c so that $f'(c) = 0$:

$$f'(c) = \frac{1}{2\sqrt{c}} - \frac{1}{3} = 0 \quad \Rightarrow \quad c = \frac{9}{4} = 2.25$$

11. Let $f(x) = x^3 - 3x + 2$ on the interval $[-2, 2]$. Verify that the function satisfies all the hypotheses of the Mean Value Theorem, then find the values of c that satisfy its conclusion.

SOLUTION: f is a polynomial, so it is continuous and differentiable at all real numbers. Now we find c so that

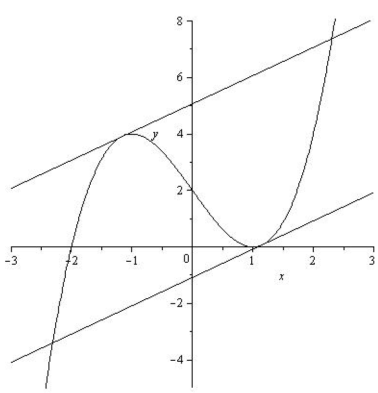
$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{4 - 0}{4} = 1$$

And $f'(c) = 3c^2 - 3$, so:

$$3c^2 - 3 = 1 \quad \Rightarrow \quad c = \pm\frac{2}{\sqrt{3}}$$

Both of these are in the interval $[-2, 2]$.

Extra: The graph is below showing the tangent lines.



12. Let $f(x) = \tan(x)$. Show that $f(0) = f(\pi)$, but there is no number c in $(0, \pi)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

SOLUTION:

$$f(0) = \tan(0) = 0 \qquad f(\pi) = \tan(\pi) = \frac{\sin(\pi)}{\cos(\pi)} = \frac{0}{-1} = 0$$

Now, $f'(c) = \sec^2(x)$. But $\sec(x) = \frac{1}{\cos(x)}$ is always greater than (or equal to) 1. That is, there is no solution to:

$$\frac{1}{\cos^2(x)} = 0$$

This does not contradict Rolle's Theorem, since $\tan(x)$ is not continuous on $[0, \pi]$ (there is a vertical asymptote at $x = \pi/2$).

13. Find the limit, if it exists.

   (a) This is a 0/0 form, so use l'Hospital's rule directly:
   $$\lim_{x\to 0} \frac{\sin^{-1}(x)}{x} \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = 1$$

   (b) This is a 0/0 form, so use l'Hospital's rule directly:
   $$\lim_{x\to 0} \frac{x3^x}{3^x - 1} = \lim_{x\to 0} \frac{3^x + x3^x \ln(3)}{3^x \ln(3)} = \lim_{x\to 0} \frac{1 + x\ln(3)}{\ln(3)} = \frac{1}{\ln(3)}$$

   (c) This is a product, $0 \cdot (-\infty)$, so convert to a fraction first:
   $$\lim_{x\to 0^+} \sin(x)\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\csc(x)} = \lim_{x\to 0^+} \frac{1/x}{-\csc(x)\cot(x)}$$
   To simplify the denominator, we see that $-\csc(x)\cot(x) = -\frac{\cos(x)}{\sin^2(x)}$, so that taking the reciprocal, and taking l'Hospital's rule one last time:
   $$= \lim_{x\to 0^+} \frac{-\sin^2(x)}{x\cos(x)} = \lim_{x\to 0^+} \frac{-2\sin(x)\cos(x)}{\cos(x) - x\sin(x)} = \frac{0}{1} = 0$$

   (d) Taking a cue from the way the problem is presented, we'll write this as a fraction taking the reciprocal of $\cot(2x)$:
   $$\lim_{x\to 0} \cot(2x)\sin(6x) = \lim_{x\to 0} \frac{\sin(6x)}{\tan(2x)} = \lim_{x\to 0} \frac{6\cos(6x)}{2\sec^2(2x)} = \frac{6\cdot 1}{2\cdot 1^2} = 3$$

   (e) We have the $f(x)^{g(x)}$ form, so use the exponential function. Algebraically,
   $$x^{\sqrt{x}} = \mathrm{e}^{\sqrt{x}\ln(x)}$$
   Therefore, taking the limit as $x \to 0^+$, we only need to consider the exponent:
   $$\lim_{x\to 0^+} \sqrt{x}\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1/2}} = \lim_{x\to 0^+} \frac{1/x}{-(1/2)x^{-3/2}} = \lim_{x\to 0^+} -2\sqrt{x} = 0$$
   Therefore,
   $$\lim_{x\to 0^+} x^{\sqrt{x}} = \mathrm{e}^0 = 1$$

   (f) The idea is the same as before, writing
   $$(4x+1)^{\cot(x)} = \mathrm{e}^{\cot(x)\ln(4x+1)}$$
   In taking the limit, we need to just consider the exponent. Before l'Hospital's rule, we need to rewrite the product as a fraction:
   $$\lim_{x\to 0^+} \cot(x)\ln(4x+1) = \lim_{x\to 0^+} \frac{\ln(4x+1)}{\tan(x)} = \lim_{x\to 0^+} \frac{\frac{4}{4x+1}}{\sec^2(x)} = \frac{4}{1} = 4$$
   So overall, $\lim_{x\to 0^+} (4x+1)^{\cot(x)} = \mathrm{e}^4$.

14. Verify the given linear approximation (for small x).

    (a) $\sqrt[4]{1+2x} \approx 1 + \frac{1}{2}x$
    SOLUTION: We show that the tangent line to $f(x) = \sqrt[4]{1+2x}$ at $x = 0$ is $1+\frac{1}{2}x$. For the tangent line, the point on the curve is $(0, 1)$ and the slope is:

    $$f'(x) = \frac{1}{2}(1+2x)^{-3/4}\Big|_{x=0} = \frac{1}{2}$$

    Therefore, the tangent line is:

    $$f(0) + f'(0)(x-0) = 1 + \frac{1}{2}x$$

    (b) $e^x \cos(x) \approx 1 + x$
    SOLUTION: We solve this the same way as the previous problem- Show that the tangent line to $f(x) = e^x \cos(x)$ at $x = 0$ is given by $1 + x$.
    For the line, the point is $(0, 1)$. The slope is:

    $$f'(x) = e^x \cos(x) - e^x \sin(x)|_{x=0} = 1 - 0 = 1$$

    Therefore, the tangent line is:

    $$f(0) + f'(0)(x-0) = 1 + x$$

15. A child is flying a kite. If the kite is 90 feet above the child's hand level and the wind is blowing it on a horizontal course at 5 feet per second, how fast is the child paying out cord when 150 feet of cord is out? (Assume that the cord forms a line- actually an unrealistic assumption).

    Note: As with all word problems, your notation may be different than mine First, draw a picture of a right triangle, with height 90, hypotenuse $y(t)$, the other leg $x(t)$. Note that these are varying until after we differentiate. So, we have that

    $$(y(t))^2 = (x(t))^2 + 90^2$$

    and

    $$2y(t)\frac{dy}{dt} = 2x(t)\frac{dx}{dt}$$

    We want to find $\frac{dy}{dt}$ when $y = 150$ and $\frac{dx}{dt} = 5$. We need to know $x(t)$, so use the first equation:

    $$150^2 - 90^2 = x^2 \Rightarrow x = 120$$

    so

    $$2 \cdot 150\frac{dy}{dt} = 2 \cdot 120 \cdot 5 \Rightarrow \frac{dy}{dt} = 4$$

16. At 1:00 PM, a truck driver picked up a fare card at the entrance of a tollway. At 2:15 PM, the trucker pulled up to a toll booth 100 miles down the road. After computing the trucker's fare, the toll booth operator summoned a highway patrol officer who issued a speeding ticket to the trucker. (The speed limit on the tollway is 65 MPH).

    (a) The trucker claimed that she hadn't been speeding. Is this possible? Explain.
    SOLUTION: Nope. Not possible. The trucker went 100 miles in 1.25 hours, which is not possible if you go (at a maximum) of 65 miles per hour (which would only get you (at a max) 81.25 miles).
    In terms of the MVT:

    $$\frac{\text{Change in Position}}{\text{Change in time}} = \frac{100}{1.25} = 80$$

    So we can guarantee that at some point in time, the trucker's speedometer read exactly 80 MPH.

(b) The fine for speeding is \$35.00 plus \$2.00 for each mph by which the speed limit is exceeded. What is the trucker's minimum fine? By the last computation, the trucker had an *average* speed of 80 mph, so we can guarantee (by the MVT) that at some point, the speedometer read exactly 80. So, this gives \$35.00 + \$2.00 (15) = \$65.00

17. Let $f(x) = \frac{1}{x}$

 (a) What does the Extreme Value Theorem (EVT) say about f on the interval $[0.1, 1]$?

 SOLUTION: Since f is continuous on this closed interval, there is a global max and global min (on the interval).

 (b) Although f is continuous on $[1, \infty)$, it has no minimum value on this interval. Why doesn't this contradict the EVT?

 SOLUTION: The EVT was stated on an interval of the form $[a, b]$, which implies that we cannot allow a, b to be infinite.

18. Let f be a function so that $f(0) = 0$ and $\frac{1}{2} \leq f'(x) \leq 1$ for all x. Explain why $f(2)$ cannot be 3 (Hint: You might use a value theorem to help).

 SOLUTION: We know that:
$$\frac{f(2) - f(0)}{2 - 0} = f'(c)$$
 for some c in $(0, 2)$. Using the restrictions on the derivative,
$$\frac{1}{2} \leq \frac{f(2)}{2} \leq 1$$
 so that $1 \leq f(2) \leq 2$.

19. Related Rates Extra Practice:

 (a) The top of a 25-foot ladder, leaning against a vertical wall, is slipping down the wall at a rate of 1 foot per second. How fast is the bottom of the ladder slipping along the ground when the bottom of the ladder is 7 feet away from the base of the wall?

 First, make a sketch of a triangle whose hypotenuse is the ladder. Let $y(t)$ be the height of the ladder with the vertical wall, and let $x(t)$ be the length of the bottom of the ladder with the vertical wall. Then
$$x^2(t) + y^2(t) = 25^2$$
 The problem can then be interpreted as: If $\frac{dy}{dt} = -1$, what is $\frac{dx}{dt}$ when $x = 7$?

 Differentiating with respect to time:
$$2x(t)\frac{dx}{dt} + 2y(t)\frac{dy}{dt} = 0$$
 we have numbers for $\frac{dy}{dt}$ and $x(t)$- we need a number for y in order to solve for $\frac{dx}{dt}$. Use the original equation, and
$$7^2 + y^2(t) = 25^2 \Rightarrow y = 24$$
 Now plug everything in and solve for $\frac{dx}{dt}$:
$$2 \cdot 7 \cdot \frac{dx}{dt} + 2 \cdot 24 \cdot (-1) = 0 \Rightarrow \frac{dx}{dt} = \frac{24}{7}$$

(b) A 5-foot girl is walking toward a 20-foot lamppost at a rate of 6 feet per second. How fast is the tip of her shadow (cast by the lamppost) moving?

Let $x(t)$ be the distance of the girl to the base of the post, and let $y(t)$ be the distance of the tip of the shadow to the base of the post. If you've drawn the right setup, you should see similar triangles...

$$\frac{\text{Hgt of post}}{\text{Hgt of girl}} = \frac{\text{Dist of tip of shadow to base}}{\text{Dist of girl to base}}$$

In our setup, this means:

$$\frac{20}{5} = \frac{y(t)}{y(t) - x(t)}$$

With a little simplification, we get:

$$3y(t) = 4x(t)$$

We can now interpret the question as asking what $\frac{dy}{dt}$ is when $\frac{dx}{dt} = -6$. Differentiating, we get

$$3\frac{dy}{dt} = 4\frac{dx}{dt}$$

so that the final answer is $\frac{dy}{dt} = -8$, which we interpret to mean that the tip of the shadow is approaching the post at a rate of 8 feet per second.

(c) Under the same conditions as above, how fast is the length of the girl's shadow changing?

Let $L(t)$ be the length of the shadow at time t. Then, by our previous setup,

$$L(t) = y(t) - x(t)$$

so $\frac{dL}{dt} = \frac{dy}{dt} - \frac{dx}{dt} = -8 - (-6) = -2$.

(d) A rocket is shot vertically upward with an initial velocity of 400 feet per second. Its height s after t seconds is $s = 400t - 16t^2$. How fast is the distance changing from the rocket to an observer on the ground 1800 feet away from the launch site, when the rocket is still rising and is 2400 feet above the ground?

We can form a right triangle, where the launch site is the vertex for the right angle. The height is $s(t)$, given in the problem, the length of the second leg is fixed at 1800 feet. Let $u(t)$ be the length of the hypotenuse. Now we have:

$$u^2(t) = s^2(t) + 1800^2$$

and we can interpret the question as asking what $\frac{du}{dt}$ is when $s(t) = 2400$. Differentiating, we get

$$2u(t)\frac{du}{dt} = 2s(t)\frac{ds}{dt} \text{ or } u(t)\frac{du}{dt} = s(t)\frac{ds}{dt}$$

To solve for $\frac{du}{dt}$, we need to know $s(t)$, $u(t)$ and $\frac{ds}{dt}$. We are given $s(t) = 2400$, so we can get $u(t)$:

$$u(t) = \sqrt{2400^2 - 1800^2} = 3000$$

Now we need $\frac{ds}{dt}$. We are given that $s(t) = 400t - 16t^2$, so $\frac{ds}{dt} = 400 - 32t$. That means we need t. From the equation for $s(t)$,

$$2400 = 400t - 16t^2 \Rightarrow -16t^2 + 400t - 2400 = 0$$

Solve this to get $t = 10$ or $t = 15$. Our rocket is on the way up, so we choose $t = 10$. Finally we can compute $\frac{ds}{dt} = 400 - 32(10) = 80$. Now,

$$3000\frac{du}{dt} = 2400(80)$$

so $\frac{du}{dt} = 64$ feet per second (at time 10).

(e) A small funnel in the shape of a cone is being emptied of fluid at the rate of 12 cubic centimeters per second (the tip of the cone is downward). The height of the cone is 20 cm and the radius of the top is 4 cm. How fast is the fluid level dropping when the level stands 5 cm above the vertex of the cone [The volume of a cone is $V = \frac{1}{3}\pi r^2 h$].

Draw a picture of an inverted cone. The radius at the top is 4, and the overall height is 20. Inside the cone, draw some water at a height of $h(t)$, with radius $r(t)$.

We are given information about the rate of change of volume of water, so we are given that $\frac{dV}{dt} = -12$. Note that the formula for volume is given in terms of r and h, but we only want $\frac{dh}{dt}$. We need a relationship between r and h...

You should see similar triangles (Draw a line right through the center of the cone. This, and the line forming the top radius are the two legs. The outer edge of the cone forms the hypotenuse).

$$\frac{\text{radius of top}}{\text{radius of water level}} = \frac{\text{overall height}}{\text{height of water}} \Rightarrow \frac{4}{r} = \frac{20}{h}$$

so that $r = \frac{h}{5}$. Substituting this into the formula for the volume will give the volume in terms of h alone:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\frac{h}{5})^2 h = \frac{1}{75}\pi h^3$$

Now,

$$\frac{dV}{dt} = \frac{3\pi}{75}h^2\frac{dh}{dt}$$

and we know $\frac{dV}{dt} = -12$, $h = 5$, so

$$\frac{dh}{dt} = \frac{-12}{\pi}$$

(f) A balloon is being inflated by a pump at the rate of 2 cubic inches per second. How fast is the diameter changing when the radius is $\frac{1}{2}$ inch?

The volume is $V = \frac{4}{3}\pi r^3$ (this formula would be given to you on an exam/quiz). If we let h be the diameter, then we know that $2r = h$, so we can make V depend on diameter instead of radius:

$$V = \frac{4}{3}\pi(\frac{h}{2})^3 = \frac{\pi}{6}h^3$$

Now, the question is asking for $\frac{dh}{dt}$ when $h = 1$, and we are given that $\frac{dV}{dt} = 2$. Differentiate, and

$$\frac{dV}{dt} = \frac{\pi}{6}3h^2\frac{dh}{dt} = \frac{\pi}{2}h^2\frac{dh}{dt}$$

so that $\frac{dh}{dt} = \frac{4}{\pi}$.

(g) A rectangular trough is 8 feet long, 2 feet across the top, and 4 feet deep. If water flows in at a rate of 2 cubic feet per minute, how fast is the surface rising when the water is 1 foot deep?

The trough is a rectangular box. Let $x(t)$ be the height of the water at time t. Then the volume of the water is:

$$V = 16x \Rightarrow \frac{dV}{dt} = 16\frac{dx}{dt}$$

Put in $\frac{dV}{dt} = 2$ to get that $\frac{dx}{dt} = \frac{1}{8}$.

(h) If a mothball (sphere) evaporates at a rate proportional to its surface area $4\pi r^2$, show that its radius decreases at a constant rate.

Let $V(t)$ be the volume at time t. We are told that

$$\frac{dV}{dt} = kA(t) = k4\pi r^2$$

We want to show that $\frac{dr}{dt}$ is constant.
By the formula for $V(t) = \frac{4}{3}\pi r^3$, we know that:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now compare the two formulas for $\frac{dV}{dt}$, and we see that $\frac{dr}{dt} = k$, which was the constant of proportionality!

(i) If an object is moving along the curve $y = x^3$, at what point(s) is the $y-$coordinate changing 3 times more rapidly than the $x-$coordinate?

Let's differentiate:

$$\frac{dy}{dt} = 3x^2 \frac{dx}{dt}$$

From this, we see that if we want $\frac{dy}{dt} = 3\frac{dx}{dt}$, then we must have $x = \pm 1$. We also could have $x = 0, y = 0$, since 0 is 3 times 0. All the points on the curve are therefore:

$$(0, 0), (-1, 1), (1, 1)$$

20. **Graphical Exercises**

The solutions to the odd problems are in the text.