Extra Practice solutions

1. For each of the following integrals, write the definition using the Riemann sum (and right endpoints), but do not evaluate them:

   (a) $\int_2^5 \sin(3x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \sin(3(2+3i/n))\,\frac{3}{n}$

   (b) $\int_1^3 \sqrt{1+x}\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+\left(1+\frac{2i}{n}\right)}\,\frac{2}{n}$

   (c) $\int_0^2 e^x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} e^{2i/n}\,\frac{2}{n}$

2. For each of the following integrals, write the definition using the Riemann sum, and then evaluate them (MUST use the limit of the Riemann sum for credit, and do not re-write them using the properties of the integral):

   (a) $\int_2^5 x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(2+\frac{3i}{n}\right)^2 \frac{3}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{12}{n}+\frac{36i}{n^2}+\frac{27i^2}{n^3}\right) =$

   $$\lim_{n\to\infty} \frac{12}{n}\sum_{i=1}^{n} 1 + \frac{36}{n^2}\sum_{i=1}^{n} i + \frac{27}{n^3}\sum_{i=1}^{n} i^2 = \lim_{n\to\infty} 12 + \frac{36}{n^2}\cdot\frac{n(n+1)}{2} + \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} =$$

   $$12+18+9=39$$

   (b) $\int_1^3 1-3x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1-3\left(1+\frac{2i}{n}\right)\right)\frac{2}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(-\frac{4}{n}-\frac{12i}{n^2}\right) = -4-6=-10$

   (c) $\int_0^5 1+2x^3\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1+2\left(\frac{5i}{n}\right)^3\right)\frac{5}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{5}{n}+\frac{10\cdot 125 i^3}{n^4}\right) =$

   $$\lim_{n\to\infty} \frac{5}{n}\sum_{i=1}^{n} 1 + \frac{10\cdot 125}{n^4}\sum_{i=1}^{n} i^3 = \lim_{n\to\infty} 5 + \frac{10\cdot 125}{n^4}\left(\frac{n(n+1)}{2}\right)^2 = 5+\frac{1250}{4} = \frac{1270}{4}$$

3. For each of the following Riemann sums, evaluate the limit by first recognizing it as an appropriate integral:

   NOTE: Multiple integrals are possible. I've written perhaps the most natural ones.

   (a) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3}{n}\right)\sqrt{1+\frac{3i}{n}} = \int_1^4 \sqrt{x}\,dx = \frac{14}{3}$

   (b) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(2+3\cdot\frac{25i^2}{n^2}\right)\left(\frac{5}{n}\right) = \int_0^5 2+3x^2\,dx = 135$

   (c) $\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(3+\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \int_3^5 \sin(x)\,dx = -\cos(5)+\cos(3)$

4. Suppose that

   $$\int_1^4 f(x)\,dx = 7 \qquad \int_2^4 f(x)\,dx = 5, \qquad \int_1^4 g(x)\,dx = 2$$

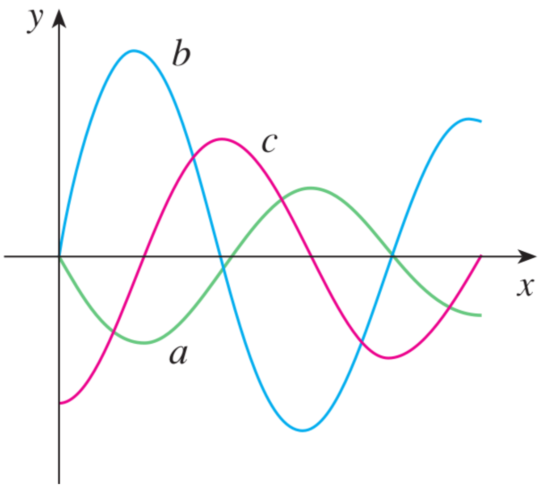
   Using only this information, compute the following:

(a) $\int_1^4 4f(x)\,dx = 4 \cdot 7 = 28$

(b) $\int_1^4 8f(x) - 7g(x)\,dx = 8\int_1^4 f(x)\,dx - 7\int_1^4 g(x)\,dx = 8 \cdot 7 - 7 \cdot 2 = 42$

(c) $\int_1^2 -f(x)\,dx = -\left(\int_1^4 f(x)\,dx - \int_2^4 f(x)\,dx\right) = -7 + 5 = -2$

5. The following figure shows the graphs of f, f' and $\int_0^x f(t)\,dt$. Identify each graph and explain your choices.



SOLUTION: It looks as though c is the derivative of a. Now, if $a = f$ and $c = f'$, then that would leave $b = \int_0^x f(t)\,dt$. However, the area function would have to start at the origin and go negative, therefore b is not the integral. Therefore, $a = \int_0^x f(t)\,dt$ and $c = f(x)$, so that leaves $b = f'(x)$.

6. Find the derivative of the function:

NOTE: For more of these, see Example 4, pg. 390, and exercises 55-59, 5.3.

(a) $F(x) = \int_0^x \frac{t^2}{1+t^2}\,dt$. $F'(x) = \frac{x^2}{1+x^2}$

(b) $F(x) = \int_x^1 \cos(t^2)\,dt$. $F'(x) = -\cos(x^2)$

(c) $F(x) = \int_{2x}^{3x+1} \frac{e^t}{t}\,dt$. $F'(x) = \frac{e^{3x+1}}{3x+1}(3) - \frac{e^{2x}}{2x}(2)$

7. Find the general indefinite integral:

(a) $\int \sqrt{x^3} + x^2 + \frac{1}{x}\,dx = \frac{2}{5}x^{5/2} + \frac{1}{3}x^3 + \ln|x| + C$

(b) $\int (u+4)(2u+1)\,du = \frac{2}{3}u^3 + \frac{9}{2}u^2 + 4u + C.$

(c) $\int \sec^2(x) + \sin(x)\,dx = \tan(x) - \cos(x) + C$

8. Evaluate the integral:

(a) $\int_1^2 \left(\frac{x}{2} - \frac{2}{x}\right)\,dx = \frac{x^2}{4} - 2\ln|x|\Big|_1^2 = \frac{3}{4} - 2\ln(2)$

(b) $\displaystyle\int_1^8 \frac{1+\sqrt[3]{u}}{\sqrt{u}}\,du = \int_1^8 u^{-1/2} + u^{-1/6}\,du = 2u^{1/2} + \frac{6}{5}u^{5/6}\Big|_1^8 = (2\sqrt{8} + \frac{6}{5}8^{5/6}) - 16/5$

(c) $\displaystyle\int_{-1}^2 x - 2|x|\,dx$

First re-write the absolute value:

$$x - 2|x| = \begin{cases} 3x & \text{if } x \le 0 \\ -x & \text{if } x < 0 \end{cases}$$

Therefore,

$$\int_{-1}^2 2 - |x|\,dx = \int_{-1}^0 3x\,dx - \int_0^3 x\,dx = -6$$

(d) $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$:

Don't forget that we can use geometry if needed. In this case,

$$y = \sqrt{1-x^2} \quad \Rightarrow \quad y^2 = 1 - x^2 \quad \text{or} \quad x^2 + y^2 = 1$$

So the graph of this function is the upper half circle. We want the area of the circle only in the first quadrant, so the answer is "the area of the circle divided by 4", which is $\pi/4$.

9. Evaluate using the given substitution:

(a) $\displaystyle\int x^3(2+x^4)^5\,dx$, $u = 2 + x^4$

SOLUTION: With the given u, then $du = 4x^3\,dx$, or $du/4 = x^3\,dx$ so that

$$\int x^3(2+x^4)^5\,dx = \int u^5\frac{du}{4} = \frac{1}{4}\frac{1}{6}u^6 + C = \frac{1}{24}(2+x^4)^6 + C$$

(b) $\displaystyle\int \frac{dt}{(1-6t)^4}$, $u = 1 - 6t$

SOLUTION: Note that $du = -6\,dt$, so

$$-\frac{1}{6}\int u^{-4}\,du = -\frac{1}{18}u^{-3} + C = -\frac{1}{18}\ln|1-6t| + C$$

(c) $\displaystyle\int \frac{\sec^2(1/x)}{x^2}\,dx$, $u = 1/x$.

SOLUTION: This time, $du = \frac{1}{x^2}\,dx$ (which is what we have already in the integral). Therefore,

$$\int \sec^2(u)\,du = \tan(u) + C = \tan(1/x) + C$$

10. Evaluate by finding a substitution:

(a) $\displaystyle\int \sin(4x)\,dx$

SOLUTION: Let $u = 4x$ so that

$$\int \sin(4x)\,dx = \frac{1}{4}\int \sin(u)\,du = -\frac{1}{4}\cos(u) + C = -\frac{1}{4}\cos(4x) + C$$

(b) $\displaystyle\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

SOLUTION: Let $u=\sqrt{x}$, so that $du=\frac{1}{2\sqrt{x}}\,dx$, or $2\,du=\frac{1}{\sqrt{x}}\,dx$. Then:

$$\frac{1}{2}\int \sin(u)\,du = -\frac{1}{2}\cos(u)+C = -\frac{1}{2}\cos(\sqrt{x})+C$$

(c) $\displaystyle\int \frac{z^2}{z^3+1}\,dz$

SOLUTION: Let $u=z^3+1$ so that $du=3z^2\,dz$, or $du/3=z^2\,dz$. Then:

$$\frac{1}{3}\frac{1}{u}\,du = \frac{1}{3}\ln|u|+C = \frac{1}{3}\ln|z^3+1|+C$$

(d) $\displaystyle\int \mathrm{e}^{2x}\,dx = \frac{1}{2}\mathrm{e}^{2x}+C$. (Let $u=2x$)