Sample Exam 2 solutions

1. Short Answer:

   (a) True or False? $\dfrac{x^2-1}{x-1} = x+1$

   SOLUTION: False, unless we restrict both expressions to all x except $x = 1$- That is, the expression on the left is not defined at $x = 1$, but the expression on the right is defined at $x = 1$.

   (b) If $f'(2)$ exists, then $\lim_{x\to 2} f(x) = f(2)$

   SOLUTION: (Typo: Should have started with "True or False") True- This says that if the derivative of f exists at $x = 2$, then f is continuous at $x = 2$ (which is a theorem we know).

   (c) If $f(x) = (2-3x)^{-1/2}$, find $f(0), f'(0)$ and $f''(0)$.

   SOLUTION: Just do the computation. $f(0) = 2^{-1/2} = \frac{1}{\sqrt{2}}$, and

$$f'(x) = -\frac{1}{2}(2-3x)^{-3/2}(-3) = \frac{3}{2(2-3x)^{3/2}} \quad \Rightarrow \quad f'(0) = \frac{3}{2^{5/2}}$$

$$f''(x) = -\frac{9}{4}(2-3x)^{-5/2}(-3) \quad \Rightarrow \quad f''(0) = \frac{27}{4\cdot 2^{5/2}} = \frac{27}{2^{9/2}}$$

   (d) Show that the $x^4 + 4x + c = 0$ has at most one solution in the interval $[-1, 1]$.

   SOLUTION: Let $f(x) = x^4 + 4x + c$. Then $f'(x) = 4x^3 + 4$, and we note that $f'(x) = 0$ only at $x = -1$. We know, by the Mean Value Theorem (or more specifically, Rolle's Theorem) that if $f(x) = 0$ twice or more in $(-1, 1]$, then $f'(x)$ would have to be zero at least once in this interval (but it is not). Therefore, f can be zero at most once in $(-1, 1]$. We can also note that if $f(x) = 0$ at $x = -1$, then again $f(x)$ cannot be zero again in the interval for the same reason (the derivative would have to be zero somewhere in $(-1, 1]$).

   NOTE: Our argument gets a little muddy if the derivative is zero at an endpoint, but basically the same one works as if the derivative was not zero at an endpoint.

2. Find dy/dx (solve for it if necessary):

   (a) $y = x\mathrm{e}^{g(\sqrt{x})}$ for some differentiable function g.

   SOLUTION: This is a product rule together with a chain rule:

$$\frac{dy}{dx} = \mathrm{e}^{g(\sqrt{x})} + x\cdot \mathrm{e}^{g(\sqrt{x})}\left(g'(\sqrt{x})\frac{1}{2\sqrt{x}}\right)$$

   (b) $y = x^2 + 4^{1/x} + \sin^{-1}(3x+1) + \sec(x^2+x)$

   SOLUTION:

$$\frac{dy}{dx} = 2x + 4^{1/2}\ln(4)\cdot\frac{-1}{x^2} + \frac{1}{\sqrt{1-(3x+1)^2}}\cdot 3 + \sec(x^2+x)\tan(x^2+x)(2x+1)$$

   (c) $x\tan(y) = y - 1$

   SOLUTION: Be sure to use the product rule on the expression to the left, and use implicit differentiation:

$$\tan(y) + x\sec^2(y)y' = y' \quad \Rightarrow \quad (x\sec^2(y) - 1)y' = -\tan(y) \quad \Rightarrow \quad y' = \frac{-\tan(y)}{x\sec^2(y) - 1}$$

   (d) $\sqrt{x} + \sqrt{y} = 1$

   SOLUTION: Same type as the previous one- Use implicit differentiation:

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}y' = 0 \quad \Rightarrow \quad y' = -\frac{\sqrt{y}}{\sqrt{x}} = -\sqrt{\frac{y}{x}}$$

3. Find $f'(x)$ directly from the definition of the derivative (using limits and without l'Hospital's rule): $f(x) = x^{-1}$

   SOLUTION:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^{-1}-(x)^{-1}}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{x-(x+h)}{x(x+h)}\right)$$

   Simplifying, we get $-1/x^2$.

   Grading Note: For partial credit, be sure you write down the definition accurately (with the limit!).

4. Find the limit if it exists. You may use any method (except for a numerical table).

   (a) $\lim_{x\to\infty} \sqrt{9x^2+x} - 3x$

   SOLUTION: Multiply by the conjugate and we'll introduce a fraction:

$$\lim_{x\to\infty} (\sqrt{9x^2+x}-3x)\frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty} \frac{9x^2+x-9x^2}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty} \frac{x/x}{(\sqrt{9x^2+x}+3x)/x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{9+\frac{1}{x}}+3} = \frac{1}{6}$$

   (b) $\lim_{x\to\pi^-} \frac{\sin(x)}{1-\cos(x)}$

   SOLUTION: Try evaluating first, and we get $0/(1 - -1) = 0/2 = 0$.

   (c) $\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)}$

   SOLUTION: Use l'Hospital's Rule:

$$\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)} = \lim_{x\to 0} \frac{1}{\frac{1}{1+(4x)^2}\cdot 4} = \lim_{x\to 0} \frac{1+16x^2}{4} = \frac{1}{4}$$

   (d) $\lim_{x\to 1} x^{1/(1-x)}$

   SOLUTION: Rewrite this using the logarithm before using l'Hospital's rule on the logarithm of the function:

$$\lim_{x\to 1} \frac{1}{1-x} \ln(x) = \lim_{x\to 1} \frac{\ln(x)}{1-x} = \lim_{x\to 1} \frac{1/x}{-1} = -1$$

   Therefore, the overall limit is e^{-1}.

5. Differentiate: $F(x) = \int_{2x}^{x^2} \mathrm{e}^{t^2}\, dt$

   SOLUTION: From the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_{h_1(x)}^{h_2(x)} f(t)\, dt = f(h_2(x))h_2'(x) - f(h_1(x))h_1'(x)$$

   so in this case,

$$F'(x) = \mathrm{e}^{x^4}(2x) - \mathrm{e}^{4x^2}(2)$$

6. Evaluate the definite integral using the definition. To get credit, you must use the limit of the Riemann sum (use right endpoints and equal widths, as is our usual practice).

$\int_2^5 x^2 + 1\, dx$

SOLUTION: The width of the rectangles is $(b-a)/n$, or in this case, $(5-2)/n = 3/n$. The i^{th} right endpoint, counting from $x = 2$ is $2 + 3i/n$, so the area of n rectangles is:

$$\sum_{i=1}^n f(x_i^*)\Delta x = \sum_{i=1}^n \left(\left(2 + \frac{3i}{n}\right)^2 + 1\right)\frac{3}{n} = \sum_{i=1}^n \left(5 + \frac{12i}{n} + \frac{9i^2}{n^2}\right)\frac{3}{n}$$

Which further simplifies to:

$$\frac{15}{n}\sum_{i=1}^n 1 + \frac{36}{n^2}\sum_{i=1}^n i + \frac{27}{n^3}\sum_{i=1}^n i^2 = \frac{15}{n}\cdot n + \frac{36}{n^2}\cdot\frac{n(n+1)}{2} + \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

Finally, we take the limit as $n \to \infty$:

$$\lim_{n\to\infty}\left(15 + \frac{18(n+1)}{n} + \frac{9(n+1)(2n+1)}{2n^2}\right) = 15 + 18 + 9 = 42$$

We can check our answer using the FTC:

$$\int_2^5 x^2 + 1\, dx = \frac{1}{3}x^3 + x\Big|_2^5 = \frac{125}{3} + 5 - \frac{8}{3} - 2 = 39 + 3 = 42$$

7. Evaluate the integral, if it exists

(a) $\displaystyle\int_1^9 \frac{\sqrt{u} - 2u^2}{u}\, du$

SOLUTION: Algebra first:

$$\int_1^9 u^{-1}(u^{1/2} - 2u^2)\, du = \int_1^9 u^{-1/2} - 2u\, du = 2u^{1/2} - u^2\Big|_1^9 = (6-81) - (2-1) = -76$$

(b) $\displaystyle\int 3^x + \frac{1}{x} + \sec^2(x)\, dx$

SOLUTION: General antiderivative here:

$$\frac{3^x}{\ln(3)} + \ln|x| + \tan(x) + C$$

(c) $\displaystyle\int_{\pi/4}^{\pi/4} \frac{t^4\tan(t)}{2+\cos(t)}\, dt$

SOLUTION: Notice that this is of the form $\displaystyle\int_a^a f(t)\, dt = 0$, so the answer is 0.

(d) $\displaystyle\int_0^3 |x^2 - 4|\, dx$

SOLUTION: We have to break up the absolute value:

$$|x^2 - 4| = \begin{cases} x^2 - 4 & \text{if } x < -2 \text{ or } x > 2 \\ -(x^2-4) & \text{if } -2 \le x \le 2 \end{cases}$$

Therefore,

$$\int_0^3 |x^2 - 4|\, dx = \int_0^2 (-x^2 + 4)\, dx + \int_2^3 x^2 - 4\, dx = \left(-\frac{1}{3}x^3 + 4x\Big|_0^2 + \left(\frac{1}{3}x^3 - 4x\Big|_2^3 = \frac{23}{3}$$

8. A water tank in the shape of an inverted cone with a circular base has a base radius of 2 meters and a height of 4 meters. If water is being pumped into the tank at a rate of 2 cubic meters per minute, find the rate at which the water level is rising when the water is 3 meters deep. ($V = \frac{1}{3}\pi r^2 h$)

   SOLUTION: If h is the height of the water, and r is the radius, the volume of water is a different than what is given for the whole cone. The volume of water is the volume of the whole cone minus the "empty" cone. That is, if h is the height of the water, then the volume is:

$$V = \frac{1}{3}\pi 2^2 \cdot 4 - \frac{1}{3}\pi r^2 (4-h)$$

   From similar triangles, we get $r = \frac{1}{2}(4-h)$, so our formula becomes:

$$V = \frac{16}{3}\pi - \frac{\pi}{12}(4-h)^3$$

   Now, treat V, h as functions of time:

$$\frac{dV}{dt} = -\frac{\pi}{12} 3(4-h)^2(-1) = \frac{\pi}{4}(4-h)^2 \frac{dh}{dt}$$

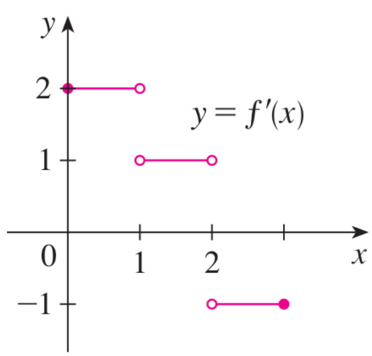
   With $dV/dt = 2$ and $h = 3$, we get: $dh/dt = 8/\pi$.

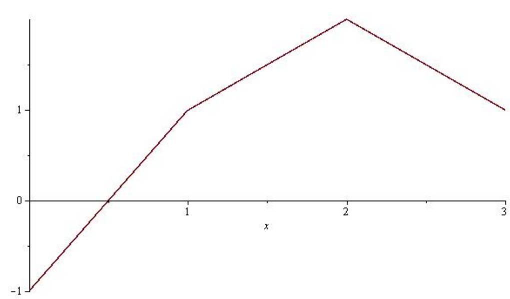
9. Explain why the following is true (if it is): The function $f(x) = \sqrt{1+2x}$ can be well approximated by $(x+5)/3$ if x is approximately 4.

   SOLTUION: Use the tangent line to $f(x)$ at $x = 4$:

$$f(4) = \sqrt{1+8} = 3 \qquad f'(4) = \frac{1}{\sqrt{1+8}} = \frac{1}{3} \quad \Rightarrow \quad L(x) = 3 + \frac{1}{3}(x-4) = \frac{x+5}{3}$$

10. The graph of $f'(x)$ is shown in the figure. Sketch the graph of f if f is continuous and $f(0) = -1$.





   SOLUTION: The antiderivative is a series of line segments- Because the question asks for the continuous one, and $f(0) = -1$, then your graph should be pretty close to the one below.

11. You're standing with Elvis (the dog) on a straight shoreline, and you throw the stick in the water. Let us label as "A" the point on the shore closest to the stick, and suppose that distance is 7 meters. Suppose that the distance from you to the point A is 10 meters. Suppose that Elvis can run at 3 meters per second, and can swim at 2 meters per second. How far along the shore should Elvis run before going in to swim to the stick, if he wants to minimize the time it takes him to get to the stick?

    SOLUTION: Recall that given distance d, rate r and time t, then $d = rt$, or $t = d/r$. If we let x be distance from point A that Elvis should travel before entering the water, then $10 - x$ is the distance that Elvis runs on the beach, and $\sqrt{x^2+49}$ is the distance Elvis has to swim. That gives the total time:

$$T(x) = \frac{\sqrt{x^2+49}}{2} + \frac{10-x}{3} \qquad 0 \le x \le 10$$

We want to minimize T on the closed interval given, so first we find the critical points, then build a table. The critical points are:

$$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2+49}} - \frac{1}{3} = 0 \quad \Rightarrow \quad 3x = 2\sqrt{x^2+49} \quad \Rightarrow \quad x = \frac{14}{\sqrt{5}}$$

(we discard the negative solution). Now build a table:

x	0	$14/\sqrt{5}$	10
T	6.833	5.94	6.10

Therefore, Elvis minimizes his overall time by first running $10 - 14/\sqrt{5} \approx 3.74$ meters, then swimming the remaining 9.39 meters for a total of 6.10 seconds.

12. Find m and b so that f is continuous and differentiable:

$$f(x) = \begin{cases} x^2 \text{ if } x \le 2 \\ mx + b \text{ if } x > 2 \end{cases}$$

SOLUTION: We note that the derivative is:

$$f'(x) = \begin{cases} 2x \text{ if } x < 2 \\ m \text{ if } x > 2 \end{cases}$$

Therefore, if we make $m = 4$, the function will be differentiable at $x = 2$. However, in order to be differentiable, the function needed to be continuous as well- Now that $m = 4$, we check to see if f is continuous at $x = 2$ by going through the definition of continuity:

- Does $f(2)$ exist? Yes. $f(2) = 2^2 = 4$.
- Does the limit exist at $x = 2$?
  - From the left: $\lim_{x \to 2^-} f(x) = 2^2 = 4$
  - From the right: $\lim_{x \to 2^+} f(x) = 4(2) + b = 8 + b$

  Therefore, the limit exists (and is $f(2)$) if $8 + b = 4$, or $b = -4$.

The function f should be:

$$f(x) = \begin{cases} x^2 \text{ if } x \le 2 \\ 4x - 4 \text{ if } x > 2 \end{cases}$$

You may note that $4x - 4$ is the tangent line to x^2 at $x = 2$ as well.

13. Find the absolute maximum and minimum of $f(x) = |x^2 - x|$ on the interval $[0, 2]$.

SOLUTION: Use the closed interval method, so we'll need the critical points. We re-write f in piecewise form:

$$f(x) = |x^2 - x| = \begin{cases} x^2 - x & \text{if } x \le 0 \text{ or } x \ge 1 \\ -x^2 + x & \text{if } 0 < x < 1 \end{cases}$$

(You can either use the graph of $x^2 - x$ or a sign chart to separate the function) Now we can differentiate as usual:

$$f'(x) = \begin{cases} 2x - 1 & \text{if } x < 0 \text{ or } x > 1 \\ -2x + 1 & \text{if } 0 < x < 1 \end{cases}$$

where we see that $f'(x)$ does not exist at $x = 0$ or $x = 1$. The derivative is zero at $x = 1/2$ as well:

x	0	1/2	1	2
f	0	1/4	0	2

Therefore, the absolute maximum is 2 (and occurs at $x = 2$), and the absolute minimum is zero (and occurs at $x = 0$ and $x = 1$).