Summary of 4.4

Suppose f and g are differentiable and $g'(x) \neq 0$ except on an open interval that contains $x = a$ (except possibly at $x = a$). Then, if

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} = \frac{\pm\infty}{\pm\infty}$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is $\pm\infty$).

A couple of cautions about l'Hospital's Rule:

- If you do not have one of the indeterminate forms, then do NOT use l'Hospital's rule.
- The conditions imply that the limit of f, g is either 0 or ∞- The method fails if the limit(s) DNE. (The example from class yesterday).

Algebra to get into the right form:

- $f(x)g(x) = f(x)/(1/g(x))$
- $y = f(x)^{g(x)} \to \ln(y) = g(x)\ln(f(x))$
  In this case, take the limit of the log of y, then at the end, exponentiate.

A couple of examples

1. $$\lim_{x\to 0}\left[\frac{1}{\ln(x+1)} - \frac{1}{x}\right] = \lim_{x\to 0} \frac{x - \ln(x+1)}{x\ln(x+1)}$$

   Now it is a form for l'Hospital's Rule:

   $$= \lim_{x\to 0} \frac{1 - 1/(x+1)}{\ln(x+1) + x\frac{1}{x+1}} = \lim_{x\to 0} \frac{x}{x+1} \cdot \frac{x+1}{(x+1)\ln(x+1) + x} = \lim_{x\to 0} \frac{x}{(x+1)\ln(x+1) + x}$$

   Use l'Hospital's rule again:

   $$\lim_{x\to 0} \frac{1}{\ln(x+1) + \frac{x+1}{x+1} + 1} = \frac{1}{2}$$

2. $$\lim_{x\to 0} \frac{\sec(x) - 1}{x^2} = \lim_{x\to 0} \frac{\sec(x)\tan(x)}{2x} = \lim_{x\to 0} \frac{\sin(x)}{2x\cos^2(x)} = \lim_{x\to 0} \frac{\cos(x)}{2\cos^2(x) - 4x\cos(x)\sin(x)} = \frac{1}{2}$$

4.7: Optimization Problems

There are two issues in Section 4.7:

- Convert the "story problem" into a mathematical problem.
- How can we optimize a function over a domain that is not closed?

For the second part, sometimes we can use the first derivative to tell us some specifics.

The First Derivative Test (for Absolute Extreme Values)

Suppose that f is continuous for all x in an interval I, and we have the following sign chart:

$f'(x)$	$+$	$-$
	$x < c$	$x > c$

Then $f(c)$ is the absolute maximum value of f on I. If we change the signs on the sign chart,

$f'(x)$	$-$	$+$
	$x < c$	$x > c$

then $f(c)$ is the absolute minimum value of f on I.

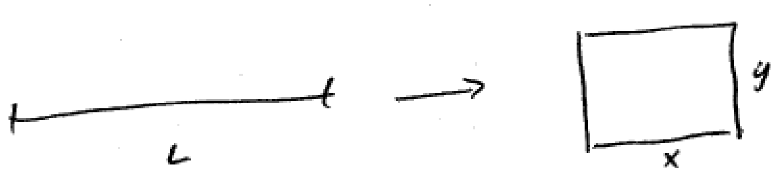
Optimization Examples

1. Suppose two nonnegative numbers are such that the first plus the square of the second is 10. Find the numbers if the sum is to be as large as possible.

   SOLUTION: First label the unknown- Let x, y be the two numbers. Then we want to:

$$\max \quad x + y \qquad \text{s.t.}$$

2. A piece of wire of length L is bent into the shape of a rectangle. What dimensions produce the rectangle with maximum area?



   SOLUTION: Draw a picture of a rectangle, and let's label the bottom as x and the side as y. Then

$$A = xy \qquad 2x + 2y = L$$

Therefore, we can make area a function of one variable:

$$A = x\left(\frac{1}{2}L - x\right) \qquad 0 \le x \le \frac{L}{2}$$

This is now a problem of maximizing a differentiable function over a closed interval:

$$A = \frac{1}{2}xL - x^2 \quad \Rightarrow \quad A' = \frac{1}{2}L - 2x = 0 \quad \Rightarrow \quad x = L/4$$

Although we don't really need it, we can construct a table:

x	0	$L/4$	$L/2$
A	0	$L^2/16$	0

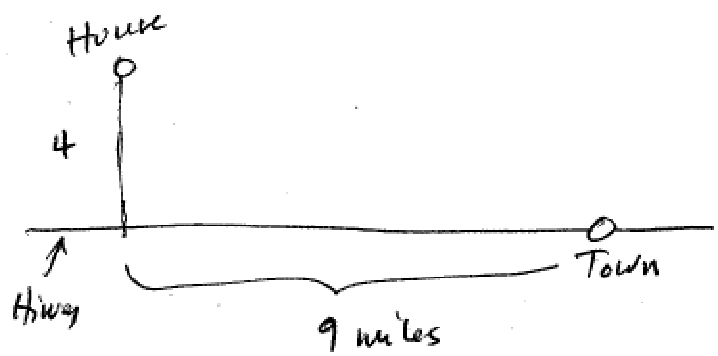
We should make a square (with side length $L/4$) to maximize the enclosed area.

NOTE: A common occurrence with these problems is to have some expression involving more than one variable, with a side equation from which we can make a substitution so that our expression has only one variable.

3. Minimize Travel Time:

RECALL: If you go at a constant rate r, then we have the ever popular formula relating distance, time and rate: $d = rt$ or $t = d/r$.

Your house is 4 miles from the highway (shortest distance), and the distance from that point on the highway to town is 9 miles (see figure). The speed on the dirt road to the highway is 20 MPH, and the speed on the highway is 60 MPH.

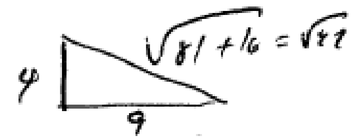


How should we build a dirt road to the highway that will minimize our total travel time to town? Before we build the answer, let's try some trial roads:

- If we build a gravel road that has the shortest distance (4 miles), the time for travel would be ($t = d/r$):

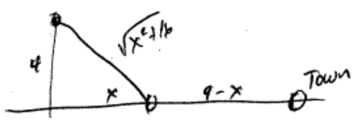
$$\frac{4}{20} + \frac{9}{60} = \frac{4+3}{20} = \frac{7}{21} \approx 0.35 \text{ hours}$$

If we go by dirt all the way, then the distance we travel is (by the Pythagorean Theorem)



- $$\sqrt{4^2+9^2} = \sqrt{97} \approx 9.8489 \text{ miles}$$

  so that the total time traveled will be: $\sqrt{97}20 \approx 0.494$ hours.
- Is there a shorter route?



Now, let x be the point on the highway which connects the dirt road, so that $0 \le x \le 9$. Then the distance traveled on the dirt road will be

$$\sqrt{x^2+16}$$

And the total time to town will be:

$$T(x) = \frac{\sqrt{x^2+16}}{20} + \frac{9-x}{60} \qquad T'(x) = \frac{1}{40}\frac{2x}{\sqrt{x^2+16}} - \frac{1}{60} = 0$$

Solving for the critical points, we get:

$$\frac{2x}{\sqrt{x^2+16}} = \frac{2}{3} \quad \Rightarrow \quad 3x = \sqrt{x^2+16} \quad \Rightarrow \quad 9x^2 = x^2+16 \quad \Rightarrow \quad x^2 = 2 \quad \Rightarrow x = \sqrt{2}$$

Build a table:

x	0	$\sqrt{2}$	9
T	0.35	0.3385	0.494

Therefore, building a dirt road that connects to the highway $\sqrt{2}$ miles down will give us the combination that gets us to town in the shortest amount of time.

4. (Econ and Agriculture)

   Experiments show that if fertilizer from N lbs of nitrogen and P lbs of phosphate is used on an acre of Kansas farmland, the number of bushels of corn per acre is:

   $$B = 8 + 0.3\sqrt{NP}$$

   Let nitrogen cost 25 cents per lb, and phosphate 20 cents per lb. A farmer intends to spend \$30 per acre on fertilizer. Which combination of nitrogen and phosphate produces the highest yield?

   SOLUTION: We want to maximize the bushels,

   $$B = 8 + 0.3\sqrt{NP}$$

   If price were no object, then our model suggests no upper limit to the amount of fertilizer! However, we have a budget. Assuming we spend exactly \$30 per acre on fertilizer,

   $$0.25N + 0.2P = 30$$

Remember what we had said before? This is a typical maximization problem, where the original function has two variables (N, P), but we have an additional constraint we can use to make the function B a function of one variable only. It doesn't matter which variable, so let's get rid of P:

$$P = 150 - 1.25N, \qquad 0 \le N \le 120$$

Substituting, we have:

$$\max_{0 \le N \le 120} 8 + 0.3\sqrt{N(150 - 1.25N)}$$

Now we have a global maximum problem on a closed interval. Find the critical points, and compute B on the CPs and endpoints:

$$\frac{dB}{dN} = 0.3 \cdot \frac{1}{2}(150N - 1.25N^2)^{-1/2}(150 - 2.5N) = 0 \quad \Rightarrow 150 = 2.5N \quad \Rightarrow \quad N = 60$$

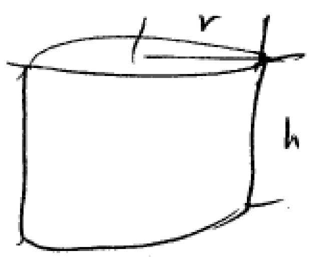
Now our table:

N	0	60	120
$B(N)$	8	28	8

We maximize the number of bushels by using 60 lbs of nitrogen, and 75 lbs of phosphate to yield 28 bushels per acre.

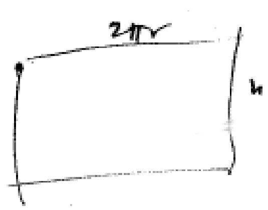
5. Design a cylindrical can that has a fixed volume of 10 ft^3 and uses the least amount of metal (include a top and bottom).

SOLUTION: First, let's get a few formulas nailed down.



A circular cylinder with radius r and height h has volume $V = \pi r^2 h$. Given this r, h we can write down the surface area:

$$A = 2\pi r^2 + 2\pi r h$$



We want to minimize the amount of material used, but note that A is a function r, h. However, we have our volume side equation that we can use to make a substitution.

$$10 = \pi r^2 h \quad \Rightarrow \quad h = \frac{10}{\pi r^2}, \quad r > 0$$

Now we can write A in terms of r, and we see that $r > 0$:

$$A = 2\pi r^2 + 2\pi r \left(\frac{10}{\pi r^2}\right) = 2\pi r^2 + \frac{20}{r}$$

Taking the derivative:

$$A' = 4\pi r - \frac{20}{r^2} = 0 \quad \Rightarrow \quad r^3 = \frac{5}{\pi} \quad \Rightarrow \quad r = \sqrt[3]{5/\pi}$$

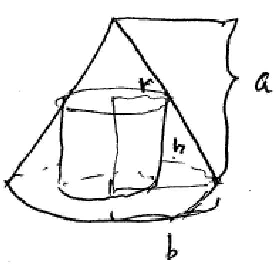
Now we need to check that this is indeed the value of r that gives us a minimum- Do this by looking at the sign of the derivative:

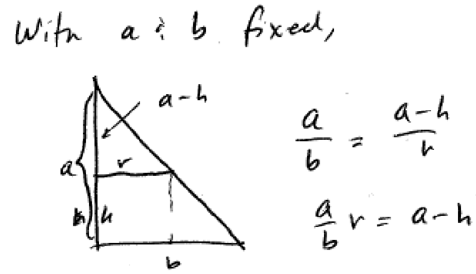
A'	$-$	$+$
	$0 < r < \sqrt[3]{5/\pi}$	$r > \sqrt[3]{5/\pi}$

Therefore, the radius should be approximately $r \approx 1.17$ and $h \approx 2.34$.

6. Find the dimensions of the right circular cylinder of greatest volume that be inscribed in a given right circular cone with radius b and height a (fixed values).

   SOLUTION: Some formulas and geometry first.





Now we want to maximize volume with the relationship as a side constraint:

$$V = \pi r^2 h \qquad h = a - \frac{a}{b} r \quad \text{and} \quad 0 \le r \le b$$

Making the substitution:

$$V = \pi r^2 \left(a - \frac{a}{b} r\right) = \pi a r^2 - \frac{\pi a}{b} r^3$$

Differentiate to find CPs:

$$\frac{dV}{dr} = 2\pi a r - 3\pi \frac{a}{b} r^2 = \pi a r \left(2 - \frac{3}{b} r\right) = 0 \quad \Rightarrow \quad r = \frac{2b}{3}, 0$$

Check CPs and endpoints- note that it doesn't really matter what the volume is at the critical point. Whatever positive number it is, that's the max.

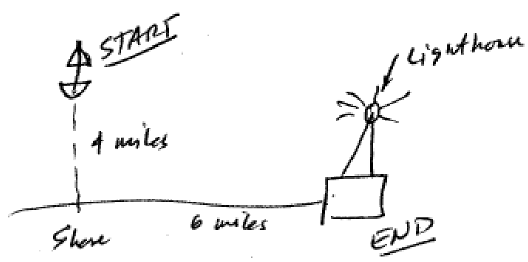
r	0	$2b/3$	b
V	0	...	0

Therefore, the cylinder with maximum volume is obtained by taking the radius and height":

$$r = \frac{2}{3} b \qquad h = \frac{a}{3}$$

7. We're on a boat 4 miles to the nearest shoreline (straight), and from that closest point on shore, a lighthous is 6 miles down. If we can row at 2 mph and walk at 3mph, at what point on the shore should we land the boat to minimize our travel time?

   SOLUTION: First draw some diagrams and determine notation.



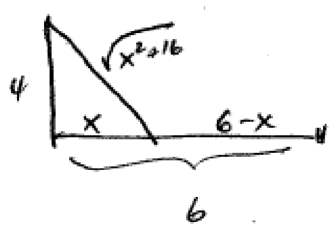
   Let x denote the distance from the shortest point to shore to where we land the boat. For example, if $x = 0$, we sail the boat straight in for 4 miles, then walk for 6 miles. This gives a travel time (use $d = rt$, or $t = d/r$) of

$$\frac{4}{2} + \frac{6}{3} = 2 + 2 = 4 \text{ hours}$$

   Similarly, if we travel by boat all the way to the lighthouse, the distance is $\sqrt{4^2 + 6^2} = \sqrt{52} \approx 7.21$, so the time:

$$\frac{\sqrt{52}}{2} + \frac{0}{3} = \sqrt{13} \approx 3.6$$

   Now, consider the figure below when we take the boat to x:



   We see that the total travel time is now

$$\frac{\sqrt{16 + x^2}}{2} + \frac{6 - x}{3}$$

   The critical point:

$$\frac{1}{4}(16 + x^2)^{-1/2}(2x) - \frac{1}{3} = 0$$

   Simplify and set to zero:

$$\frac{x}{2\sqrt{16 + x^2}} = \frac{1}{3} \quad \Rightarrow \quad 3x = 2\sqrt{16 + x^2} \quad \Rightarrow \quad 9x^2 = 4(16 + x^2)$$

   so we get $x = 8/\sqrt{5}$. Finally, we build our table and take the max:

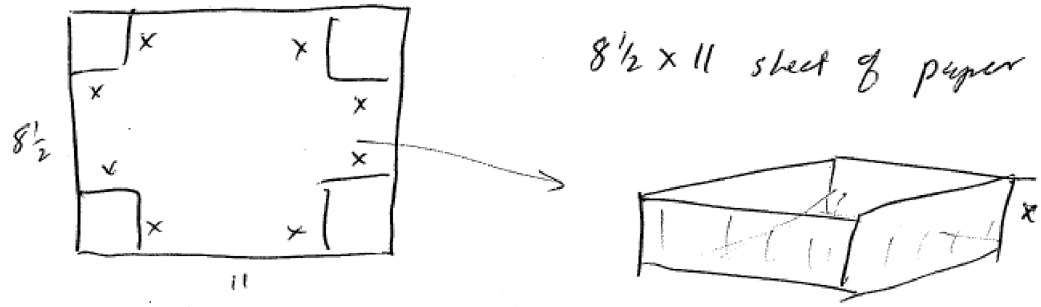
x	0	$8/\sqrt{5}$	6
time	4	$2 + 2\sqrt{5}/3 \approx 3.5$	3.6

   For the shortest travel time, land the boat at $x = 8/\sqrt{5}$ units down the shore, then walk the rest of the way in approximately 3.5 hours.

8. (The open box problem)

   Suppose we're given a sheet of paper from which we cut out four squares, one square from each corner. We fold the edges to make a box with no top. How big should I cut the squares to maximize the volume of the box?

   SOLUTION: First draw a sketch and set the notation.



We can maximize the volume of such a box in a straightfoward manner:

$$V = (11 - 2x)(8.5 - 2x)x = 4x^3 - 39x^2 + 93\frac{1}{2}x, \qquad 0 \le x \le 4\frac{1}{4}$$

The volume is zero at both endpoints, so as long as we have some non-zero volume at the critical point, we've found the max. Use the quadratic formula to solve for x:

$$\frac{dV}{dx} = 12x^2 - 78x + \frac{187}{2} = 0 \quad \Rightarrow \quad x = 1.58, 4.91$$

We only keep $x = 1.58$ since the other value is outside our interval, and this point is the maximizer with $V \approx 66.14$.

9. Find the maximum volume of a box, open at the top, with a square base and is composed of 600 square inches of material.

   SOLUTION: Draw a sketch, and let the box have base dimensions $s \times s$ with height h. Then we

$$\max V = s^2 h \qquad A = s^2 + 4sh = 600$$

   Again, we have a function of two variables, but we use the side constraint to make the function a function of one variable. Writing h in terms of s,

$$h = \frac{600 - s^2}{4s}$$

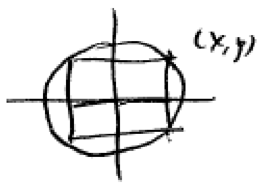
   Now our maximization problem becomes:

$$V = s^2 \left(\frac{600 - s^2}{4s} \right) = 150s - \frac{1}{4}s^3, \quad 0 \le s \le \sqrt{600}$$

$$\frac{dV}{ds} = 150 - \frac{3}{4}s^2 = 0 \quad \Rightarrow \quad s^2 = 200 \quad \Rightarrow \quad s = 10\sqrt{2}$$

   Using a table, we see that the volume at $s = 0$ and $s = \sqrt{600}$ is 0, so the value at our only critical point must be the absolute maximum.

10. Find the maximum area of a rectangle inscribed in a circle of radius 1.



Start by sketching this out and getting some relevant formulas. It is important to note that the corner of the rectangle sits at the point (x, y) on the unit circle, so that

$$x^2 + y^2 = 1$$

The dimensions of the rectangle are then $2x \times 2y$, and we have our maximization problem.

We want to

$$\max A = 4xy \qquad \text{such that } x^2 + y^2 = 1$$

Use the side constraint to make the function a function of one variable.

$$A = 4x\sqrt{1 - x^2}, \qquad 0 \leq x \leq 1$$

Differentiate to find CPs:

$$\frac{dA}{dx} = 4\sqrt{1 - x^2} + 4x(1/2)(1 - x^2)^{-1/2}(-2x) = \frac{4(1 - x^2) - 4x^2}{\sqrt{1 - x^2}} = \frac{4(1 - 2x^2)}{\sqrt{1 - x^2}} = 0$$

Therefore, noting the area at the endpoints is zero,

$$x = \frac{1}{\sqrt{2}}, \quad y = \frac{1}{\sqrt{2}} \text{ gives the max area of } 2$$