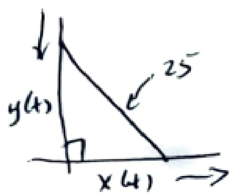
Miscellaneous Related Rates: Solutions

To do these problems, you may need to use one or more of the following: The Pythagorean Theorem, Similar Triangles, Proportionality (A is proportional to B means that $A = kB$, for some constant k).

1. The top of a 25-foot ladder, leaning against a vertical wall, is slipping down the wall at a rate of 1 foot per second. How fast is the bottom of the ladder slipping along the ground when the bottom of the ladder is 7 feet away from the base of the wall?



First, make a sketch of a triangle whose hypotenuse is the ladder. Let $y(t)$ be the height of the ladder with the vertical wall, and let $x(t)$ be the length of the bottom of the ladder with the vertical wall. Then

$$x^2(t) + y^2(t) = 25^2$$

The problem can then be interpreted as: If $\frac{dy}{dt} = -1$, what is $\frac{dx}{dt}$ when $x = 7$?

Differentiating with respect to time:

$$2x(t)\frac{dx}{dt} + 2y(t)\frac{dy}{dt} = 0$$

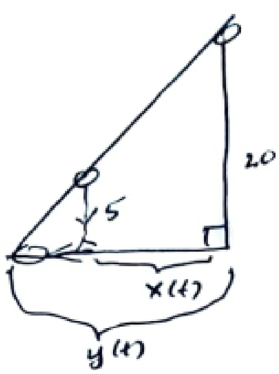
we have numbers for $\frac{dy}{dt}$ and $x(t)$- we need a number for y in order to solve for $\frac{dx}{dt}$. Use the original equation, and

$$7^2 + y^2(t) = 25^2 \Rightarrow y = 24$$

Now plug everything in and solve for $\frac{dx}{dt}$:

$$2 \cdot 7 \cdot \frac{dx}{dt} + 2 \cdot 24 \cdot (-1) = 0 \Rightarrow \frac{dx}{dt} = \frac{24}{7}$$

2. A 5-foot girl is walking toward a 20-foot lamppost at a rate of 6 feet per second. How fast is the tip of her shadow (cast by the lamppost) moving?



Let $x(t)$ be the distance of the girl to the base of the post, and let $y(t)$ be the distance of the tip of the shadow to the base of the post. If you've drawn the right setup, you should see similar triangles...

$$\frac{\text{Hgt of post}}{\text{Hgt of girl}} = \frac{\text{Dist of tip of shadow to base}}{\text{Dist of girl to base}}$$

In our setup, this means:

$$\frac{20}{5} = \frac{y(t)}{y(t) - x(t)}$$

With a little simplification, we get:

$$3y(t) = 4x(t)$$

We can now interpret the question as asking what $\frac{dy}{dt}$ is when $\frac{dx}{dt} = -6$. Differentiating, we get

$$3\frac{dy}{dt} = 4\frac{dx}{dt}$$

so that the final answer is $\frac{dy}{dt} = -8$, which we interpret to mean that the tip of the shadow is approaching the post at a rate of 8 feet per second.

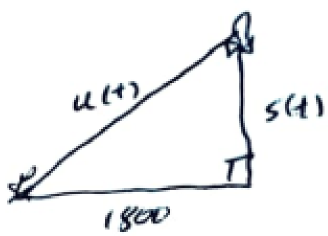
3. Under the same conditions as above, how fast is the length of the girl's shadow changing?

   Let $L(t)$ be the length of the shadow at time t. Then, by our previous setup,

$$L(t) = y(t) - x(t)$$

   so $\frac{dL}{dt} = \frac{dy}{dt} - \frac{dx}{dt} = -8 - (-6) = -2$.

4. A rocket is shot vertically upward with an initial velocity of 400 feet per second. Its height s after t seconds is $s = 400t - 16t^2$. How fast is the distance changing from the rocket to an observer on the ground 1800 feet away from the launch site, when the rocket is still rising and is 2400 feet above the ground?



   We can form a right triangle, where the launch site is the vertex for the right angle. The height is $s(t)$, given in the problem, the length of the second leg is fixed at 1800 feet. Let $u(t)$ be the length of the hypotenuse. Now we have:

$$u^2(t) = s^2(t) + 1800^2$$

   and we can interpret the question as asking what $\frac{du}{dt}$ is when $s(t) = 2400$. Differentiating, we get

$$2u(t)\frac{du}{dt} = 2s(t)\frac{ds}{dt} \text{ or } u(t)\frac{du}{dt} = s(t)\frac{ds}{dt}$$

   To solve for $\frac{du}{dt}$, we need to know $s(t)$, $u(t)$ and $\frac{ds}{dt}$. We are given $s(t) = 2400$, so we can get $u(t)$:

$$u(t) = \sqrt{2400^2 - 1800^2} = 3000$$

   Now we need $\frac{ds}{dt}$. We are given that $s(t) = 400t - 16t^2$, so $\frac{ds}{dt} = 400 - 32t$. That means we need t. From the equation for $s(t)$,

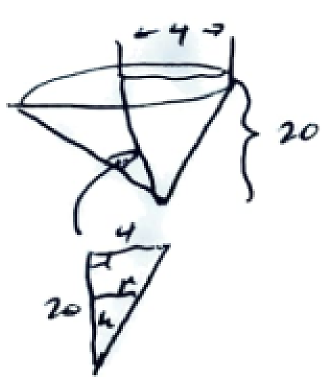
$$2400 = 400t - 16t^2 \Rightarrow -16t^2 + 400t - 2400 = 0$$

   Solve this to get $t = 10$ or $t = 15$. Our rocket is on the way up, so we choose $t = 10$. Finally we can compute $\frac{ds}{dt} = 400 - 32(10) = 80$. Now,

$$3000\frac{du}{dt} = 2400(80)$$

   so $\frac{du}{dt} = 64$ feet per second (at time 10).

5.



   A small funnel in the shape of a cone is being emptied of fluid at the rate of 12 cubic centimeters per second (the tip of the cone is downward). The height of the cone is 20 cm and the radius of the top is 4 cm. How fast is the fluid level dropping when the level stands 5 cm above the vertex of the cone [The volume of a cone is $V = \frac{1}{3}\pi r^2 h$]. Draw a picture of an inverted cone. The radius at the top is 4, and the overall height is 20. Inside the cone, draw some water at a height of $h(t)$, with radius $r(t)$.

   We are given information about the rate of change of volume of water, so we are given that $\frac{dV}{dt} = -12$. Note that the formula for volume is given in terms of r and h, but we only want $\frac{dh}{dt}$. We need a relationship between r and h...

You should see similar triangles (Draw a line right through the center of the cone. This, and the line forming the top radius are the two legs. The outer edge of the cone forms the hypotenuse).

$$\frac{\text{radius of top}}{\text{radius of water level}} = \frac{\text{overall height}}{\text{height of water}} \Rightarrow \frac{4}{r} = \frac{20}{h}$$

so that $r = \frac{h}{5}$.

Substituting this into the formula for the volume will give the volume in terms of h alone:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\frac{h}{5})^2 h = \frac{1}{75}\pi h^3$$

Now,

$$\frac{dV}{dt} = \frac{3\pi}{75} h^2 \frac{dh}{dt}$$

and we know $\frac{dV}{dt} = -12$, $h = 5$, so

$$\frac{dh}{dt} = \frac{-12}{\pi}$$

6. A balloon is being inflated by a pump at the rate of 2 cubic inches per second. How fast is the diameter changing when the radius is $\frac{1}{2}$ inch?

   The volume is $V = \frac{4}{3}\pi r^3$ (this formula would be given to you on an exam/quiz). If we let h be the diameter, then we know that $2r = h$, so we can make V depend on diameter instead of radius:

   $$V = \frac{4}{3}\pi(\frac{h}{2})^3 = \frac{\pi}{6}h^3$$

   Now, the question is asking for $\frac{dh}{dt}$ when $h = 1$, and we are given that $\frac{dV}{dt} = 2$. Differentiate, and

   $$\frac{dV}{dt} = \frac{\pi}{6} 3h^2 \frac{dh}{dt} = \frac{\pi}{2} h^2 \frac{dh}{dt}$$

   so that $\frac{dh}{dt} = \frac{4}{\pi}$.

7. A particle moves on the hyperbola $x^2 - 18y^2 = 9$ in such a way that its y coordinate increases at a constant rate of 9 units per second. How fast is the $x-$coordinate changing when $x = 9$?

   In this example, we don't need any labels. The question is to find $\frac{dx}{dt}$ when $\frac{dy}{dt} = 9$. Differentiate to get:

   $$2x\frac{dx}{dt} - 36y\frac{dy}{dt} = 0$$

   We're going to need the y value when $x = 9$, so go back to the original equation:

   $$9^2 - 18y^2 = 9 \Rightarrow y = \pm 2$$

   so we need to consider 2 $y-$values. Putting these into our derivative, we get:

   $$2 \cdot 9 \cdot \frac{dx}{dt} - 36 \cdot 2 \cdot 9 = 0$$

   so that $\frac{dx}{dt} = 36$. Put in $y = -2$ to get the second value of $\frac{dx}{dt} = -36$

8. An object moves along the graph of $y = f(x)$. At a certain point, the slope of the curve is $\frac{1}{2}$ and the $x-$coordinate is decreasing at 3 units per second. At that point, how fast is the $y-$coordinate changing?

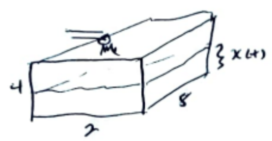
The key point here is that we want to think about both x and y as functions of t, so that when we differentiate, we get (use chain rule):

$$\frac{dy}{dt} = f'(x)\frac{dx}{dt}$$

The slope of the curve at a certain point is $f'(x) = \frac{1}{2}$, and $\frac{dx}{dt} = -3$, so we can plug these values in to get the change in y (with respect to time):

$$\frac{dy}{dt} = \frac{1}{2} \cdot (-3) = \frac{-3}{2}$$

9. A rectangular trough is 8 feet long, 2 feet across the top, and 4 feet deep. If water flows in at a rate of 2 cubic feet per minute, how fast is the surface rising when the water is 1 foot deep?



The trough is a rectangular box. Let $x(t)$ be the height of the water at time t. Then the volume of the water is:

$$V = 16x \Rightarrow \frac{dV}{dt} = 16\frac{dx}{dt}$$

Put in $\frac{dV}{dt} = 2$ to get that $\frac{dx}{dt} = \frac{1}{8}$.

10. If a mothball (sphere) evaporates at a rate proportional to its surface area $4\pi r^2$, show that its radius decreases at a constant rate.

Let $V(t)$ be the volume at time t. We are told that

$$\frac{dV}{dt} = kA(t) = k4\pi r^2$$

We want to show that $\frac{dr}{dt}$ is constant.

By the formula for $V(t) = \frac{4}{3}\pi r^3$, we know that:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now compare the two formulas for $\frac{dV}{dt}$, and we see that $\frac{dr}{dt} = k$, which was the constant of proportionality!

11. If an object is moving along the curve $y = x^3$, at what point(s) is the $y-$coordinate changing 3 times more rapidly than the $x-$coordinate?

Let's differentiate:

$$\frac{dy}{dt} = 3x^2\frac{dx}{dt}$$

From this, we see that if we want $\frac{dy}{dt} = 3\frac{dx}{dt}$, then we must have $x = \pm 1$. We also could have $x = 0, y = 0$, since 0 is 3 times 0. All the points on the curve are therefore:

$$(0,0), (-1,1), (1,1)$$