SAMPLE EXAM 1 SOLUTIONS and COMMENTS

1. Give the definition: $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

   Grading note: The limit is a *crucial* part of this definition. Also, remember to keep the arguments consistent- For example, if you're defining $f'(x)$, don't use $f(a+h)$ in the difference quotient.

2. State the Fundamental Theorem of Calculus. Let f be continuous on $[a, b]$.

   - If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
     Note that this says that g is a specific antiderivative.
   - Let F be any antiderivative of f. Then $\int_a^b f(x)\,dx = F(b) - F(a)$

3. True or False (and give a short reason):

   (a) If f is continuous at $x = a$, then f is differentiable at $x = a$.
   SOLUTION: False. For example, $f(x) = |x|$ is continuous at $x = 0$, but the function is not differentiable at $x = 0$.

   (b) If $3 \le f(x) \le 5$ then $6 \le \int_1^3 f(x)\,dx \le 10$
   SOLUTION: True, since

   $$3 \le f(x) \le 5 \quad \Rightarrow \quad 3\cdot(3-1) \le \int_1^3 f(x)\,dx \le 5\cdot(3-1)$$

   (c) All continuous functions have antiderivatives.
   SOLUTION: True. If $f(x)$ is continuous, then $g(x) = \int_a^x f(t)\,dt$ is an antiderivative by the FTC.

   (d) $\int_{-1}^2 -x^{-2}\,dx = \cdot$
   SOLUTION: False. Since $-1/x^2$ is NOT continuous on $[-1, 2]$, we cannot use the FTC to evaluate the integral.

4. Find $f'(x)$ directly from the definition of the derivative (using limits and without using l'Hospital's rule):

   $$f(x) = \sqrt{1+x}$$

   SOLUTION: Once we set up the difference quotient, multiply by the conjugate:

   $$\lim_{h\to 0} \frac{\sqrt{1+x+h}-\sqrt{1+x}}{h} \cdot \frac{\sqrt{1+x+h}+\sqrt{1+x}}{\sqrt{1+x+h}+\sqrt{1+x}} =$$

   $$\lim_{h\to 0} \frac{(1+x+h)-(1+x)}{h(\sqrt{1+x+h}+\sqrt{1+x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{1+x+h}+\sqrt{1+x})} = \frac{1}{2\sqrt{1+x}}$$

5. Derive the formula for the derivative of $y = \sin^{-1}(x)$.

   SOLUTION: First we re-write the function so we can implicitly differentiate it.

   $$\sin(y) = x$$

   This corresponds to a right triangle with an angle y, opposite length x, hypotenuse 1, and adjacent side $\sqrt{1-x^2}$ (by the Pythagorean Theorem). Now differentiate and convert back to x:

   $$\cos(y)y' = 1 \quad \Rightarrow \quad y' = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$$

6. Find dy/dx (solve for it, if necessary):

   (a) $y = \sin^3(x^2+1) + \tan^{-1}(x)$
   SOLUTION: Note the triple composition, so we'll use the chain rule:
   $$\frac{dy}{dx} = 3\sin^2(x^2+1)\cos(x^2+1)\cdot 2x + \frac{1}{x^2+1}$$

   (b) $y = 3^{1/x} + \sec(x)$
   SOLUTION: $y' = 3^{1/x}\ln(3)\cdot(-x^{-2}) + \sec(x)\tan(x)$

   (c) $\sqrt{x+y} = 4xy$
   SOLUTION: Remember to use the product rule for the right side of the equation-
   $$\frac{1+y'}{2\sqrt{x+y}} = 4y + 4xy'$$
   Solving for y', break up the fraction on the left and group y' terms together:
   $$y'\left(\frac{1}{2\sqrt{x+y}} - 4x\right) = -\frac{1}{2\sqrt{x+y}} + 4y$$
   Let's make these single fractions to make the solution clearer:
   $$y'\left(\frac{1-8x\sqrt{x+y}}{2\sqrt{x+y}}\right) = \frac{-1+8y\sqrt{x+y}}{2\sqrt{x+y}} \quad\Rightarrow\quad y' = \frac{-1+8y\sqrt{x+y}}{1-8x\sqrt{x+y}}$$
   Grading note: You should end up with a single fraction, and not a compound fraction. You may use terms like $(x+y)^{-1/2}$ in your fraction, although it would be nice to clean up the terms.

7. Find the limit, if it exists (you may use any technique from class):

   (a) $\lim_{x\to 0} \frac{1-e^{-2x}}{\sec(x)} = \frac{0}{1} = 0$

   (b) $\lim_{x\to 4^+} \frac{x-4}{|x-4|} = 1$ NOTE: You should rewrite first as $\frac{x-4}{|x-4|} = \begin{cases} 1 & \text{if } x > 4 \\ -1 & \text{if } x < 4 \end{cases}$

   (c) $\lim_{x\to -\infty} \sqrt{\frac{2x^2-1}{x+8x^2}}$
   SOLUTION: Divide numerator and denominator by x, and since $x < 0$, we will use the substitution $x = -\sqrt{x^2}$ to simplify. That is:
   $$\lim_{x\to-\infty} \sqrt{\frac{2x^2-1}{x+8x^2}} = \lim_{x\to-\infty} \frac{\sqrt{2x^2-1}/x}{\sqrt{x+8x^2}/x} = \lim_{x\to-\infty} \frac{-\sqrt{\frac{2x^2-1}{x^2}}}{-\sqrt{\frac{x+8x^2}{x^2}}} = \lim_{x\to-\infty} \frac{\sqrt{2-\frac{1}{x^2}}}{\sqrt{\frac{1}{x}+8}} = \frac{1}{2}$$
   GRADING notes: Even though in this case it turned out that the negative signs canceled, that doesn't always happen. Also, don't leave your answer as $\sqrt{2/8}$- Go ahead and simplify that to 1/2.

   (d) $\lim_{x\to 0^+} x^x$
   For this, we'll need l'Hospital's rule. We need to manipulate the expression first.
   $$x^x = e^{\ln(x^x)} = e^{x\ln(x)}$$
   Therefore, we'll take the limit of the exponent first:
   $$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0$$
   Overall, the limit is e^0, or 1.

8. Evaluate the Riemann sum by first writing it as an appropriate definite integral: $\lim_{n\to\infty}\sum_{n=1}^{\infty}\frac{3}{n}\sqrt{1+\frac{3i}{n}}$.

   We see that a natural choice for $(b-a)/n$ is $3/n$, and $a+i\frac{b-a}{n}=1+\frac{3i}{n}$. In that case, $a=1, b=1+3=4$, and $f(x)=\sqrt{x}$. In that case,

$$\lim_{n\to\infty}\sum_{n=1}^{\infty}\frac{3}{n}\sqrt{1+\frac{3i}{n}}=\int_1^4\sqrt{x}\,dx=\frac{2}{3}x^{3/2}\Big|_1^4=\frac{2}{3}(2^3-1^{3/2})=\frac{14}{3}$$

   Notice that there are an infinite number of equivalent choices here. For example,

$$\int_0^3\sqrt{1+x}\,dx,\qquad\int_2^5\sqrt{x-1}\,dx,\qquad\int_3^6\sqrt{x-2}\,dx,\qquad\text{etc.}$$

9. Differentiate: $F(x)=\int_{\sqrt{x}}^{x^2}\frac{t}{1+t}\,dt$

   $F'(x)=\frac{x^2}{1+x^2}(2x)-\frac{\sqrt{x}}{1+\sqrt{x}}\frac{1}{2\sqrt{x}}$

   This comes from the generalization to the FTC. If $g(x)=\int_a^x f(t)\,dt$, then

$$F(x)=\int_{h_1(x)}^{h_2(x)}f(t)\,dt=g(h_1(x))-g(h_2(x))\quad\Rightarrow$$

$$F'(x)=g'(h_2(x))h_2'(x)-g'(h_1(x))h_1'(x)=f(h_2(x))h_2'(x)-f(h_1(x))h_1'(x)$$

10. Use the definition to evaluate $\int_0^3 1+3x\,dx$

$$\int_0^3 1+3x\,dx=\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+3\left(\frac{3i}{n}\right)\right)\frac{3}{n}=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}+\frac{27i}{n^2}=\lim_{n\to\infty}\left(\frac{3}{n}\sum_{i=1}^{n}1+\frac{27}{n^2}\sum_{i=1}^{n}i\right)=$$

$$\lim_{n\to\infty}\left(\frac{3}{n}\cdot n+\frac{27}{n^2}\frac{n(n+1)}{2}\right)=3+\frac{27}{2}=\frac{33}{2}$$

   As a check, we can use FTC: $\int_0^3 1+3x\,dx=(x+(3/2)x^2|_0^3=3+\frac{27}{2}=\frac{33}{2}$

11. Evaluate, or find the general indefinite integral.

   (a) $\int\sqrt{x^3}+\frac{1}{x^2+1}\,dx=\frac{2}{5}x^{5/2}+\tan^{-1}(x)+C$

   (b) $\int_{-1}^{1}t(1-t)\,dt=\frac{1}{2}t^2-\frac{1}{3}t^3\Big|_{-1}^{1}=\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{1}{2}+\frac{1}{3}\right)=-\frac{2}{3}$

   (c) $\int_0^1 5x-5^x\,dx=\frac{5}{2}x^2-\frac{5^x}{\ln(5)}\Big|_0^1=\left(\frac{5}{2}-\frac{5}{\ln(5)}\right)-\left(0-\frac{1}{\ln(5)}\right)=\frac{5}{2}-\frac{4}{\ln(5)}$

12. Evaluate:

   (a) $\int_0^1\frac{d}{dx}\left(e^{\tan^{-1}(x)}\right)\,dx=e^{\tan^{-1}(1)}-e^{\tan^{-1}(0)}$

   Explanation: This is of the form $\int_0^1 f'(x)\,dx=f(1)-f(0)$, where $f(x)=e^{\tan^{-1}(x)}$.

(b) $\dfrac{d}{dx}\displaystyle\int_0^1 \mathrm{e}^{\tan^{-1}(x)}\,dx = 0$

Explanation: Recall that $\int_0^1 f(x)\,dx$ is a number, and the derivative of a constant is zero.

(c) $\dfrac{d}{dx}\displaystyle\int_0^x \mathrm{e}^{\tan^{-1}(t)}\,dt = \mathrm{e}^{\tan^{-1}(x)}$

Explanation: This is the FTC- If $g(x) = \int_0^x f(t)\,dt$, then $g'(x) = f(x)$.

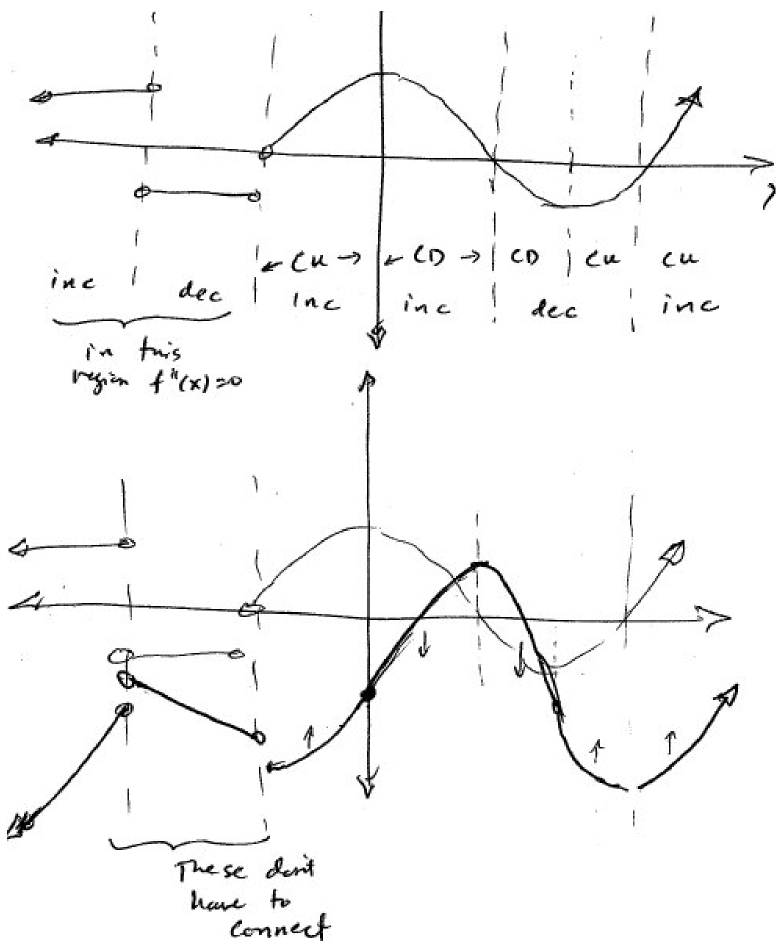
13. Given the graph of the derivative, $f'(x)$, below, answer the following questions:

(a) Find all intervals on which f is increasing.

(b) Find all intervals on which f is concave up.

(c) Sketch a possible graph of f if we require that $f(0) = -1$.

SOLUTION:



14. A rectangle is to be inscribed between the $x-$axis and the upper part of the graph of $y = 8 - x^2$ (symmetric about the $y-$axis). For example, one such rectangle might have vertices: $(1,0), (1,7), (-1,7), (-1,0)$ which would have an area of 14. Find the dimensions of the rectangle that will give the largest area.

SOLUTION: Try drawing a picture first: The parabola opens down, goes through the $y-$intercept at 8, and has $x-$intercepts of $\pm\sqrt{8}$.

Now, let x be as usual, so that the full length of the base of the rectangle is $2x$. Then the height is y, or $8 - x^2$. Therefore, the area of the rectangle is:

$$A = 2xy = 2x(8 - x^2) = 16x - 2x^3$$

and $0 \leq x \leq \sqrt{8}$. We see that the area will be zero at the endpoints, so we expect a maximum at the critical point inside the interval:

$$\frac{dA}{dx} = 16 - 6x^2$$

so the critical points are $x = \pm\sqrt{8/3}$, of which only the positive one is in our interval. So the dimensions of the rectangle are as follows (which give the maximum area of approx. 17.4):

$$2x = 2\sqrt{8}3 \qquad y = 8 - \frac{8}{3} = \frac{16}{3}$$

15. Find all values of c and d so that f is continuous at all real numbers:

$$f(x) = \begin{cases} 2x^2 - 1 & \text{if } x < 0 \\ cx + d & \text{if } 0 \leq x \leq 1 \\ \sqrt{x+3} & \text{if } x > 1 \end{cases}$$

Be sure it is clear from your work that you understand the definition of continuity.

SOLUTION: First, we note that f is continuous for all values of c, d if we remove $x = 0$ and $x = 1$ from the domain, so those are the only points we need to check. We'll check $x = 0$ first:

- $f(0)$ exists? Yes: $f(0) = 0 + d = d$ (Exists for all choices of c, d).
- Does the limit exist at $x = 0$? Check both directions:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 2x^2 - 1 = -1 \qquad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} cx + d = d$$

  For the limit to exist overall, we must have: $d = -1$ (which will also make the limit equal to $f(0)$).

Check $x = 1$ now (and we'll go ahead and replace d with -1):

- $f(1)$ exists? Yes: $f(1) = c - 1$ (Exists for all choices of c, d).
- Does the limit exist at $x = 1$? Check both directions:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} cx - 1 = c - 1 \qquad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{x+3} = 2$$

  For the limit to exist overall, we must have: $c - 1 = 2$, or $c = 3$

Therefore, if $c = 3$ and $d = -1$, f will be continuous at every x.