Exam 2 Review Questions

Sections 3.1-3.6 with 3.9 will be on the exam... Also, don't forget to go over old quizzes and homework.

If you would like extra problems, the chapter review in the text is excellent.

1. Short Answer:
   (a) How do we define the inverse sine function? (Pay attention to the domain, range and whether the domain, range are angle measures or the ratios of a triangle).
   (b) What is a *normal line*?
   (c) How do we differentiate a function that involves the absolute value?
   (d) $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}$ =? (You don't need to justify)
   (e) $\lim_{\theta\to 0} \frac{\cos(\theta)-1}{\theta}$ =? (You don't need to justify)
   (f) $\lim_{\theta\to 0} \frac{\tan(3t)}{\sin(2t)}$ =? (Do provide details)
2. Prove the Reciprocal Rule using the Product Rule (Hint: Start with $f(x) = 1/g(x)$, then write $f(x)g(x) = 1$).
3. Prove the Quotient Rule using the Product and Reciprocal Rules:
4. True or False, and explain:
   (a) The derivative of a polynomial is a polynomial.
   (b) If f is differentiable, then $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$
   (c) The derivative of $y = \sec^{-1}(x)$ is the derivative of $y = \cos(x)$.
   (d) $\frac{d}{dx}(10^x) = x10^{x-1}$
   (e) If $y = \ln|x|$, then $y' = \frac{1}{x}$
   (f) The equation of the tangent line to $y = x^2$ at $(1, 1)$ is: $y - 1 = 2x(x - 1)$
   (g) If $y = e^2$, then $y' = 2e$
   (h) If $y = |x^2 - x|$, then $y' = |2x - 1|$.
   (i) If $y = ax + b$, then $\frac{dy}{da} = x$
5. Find the equation of the tangent line to $x^3 + y^3 = 3xy$ at the point $(\frac{3}{2}, \frac{3}{2})$.
6. If $f(0) = 0$, and $f'(0) = 2$, find the derivative of $f(f(f(f(x))))$ at $x = 0$.
7. If $f(x) = 2x + \mathrm{e}^x$, find the equation of the tangent line **to the inverse** of f at $(1, 0)$. HINT: Do not try to compute f^{-1} algebraically.
8. Derive the formula for the derivative of $y = \csc^{-1}(x)$ using implicit differentiation.

9. Find the equation of the tangent line to $\sqrt{y} + xy^2 = 5$ at the point $(4, 1)$.

10. If $s^2t + t^3 = 1$, find $\frac{dt}{ds}$ and $\frac{ds}{dt}$. Are these expressions related to the derivative of a function and the derivative of its inverse?

11. If $y = x^3 - 2$ and $x = 3z^2 + 5$, then find $\frac{dy}{dz}$.

12. A space traveler is moving from left to right along the curve $y = x^2$. When she shuts off the engines, she will go off along the tangent line at that point. At what point should she shut off the engines in order to reach the point $(4, 15)$?

13. A particle moves in the plane according to the law $x = t^2 + 2t$, $y = 2t^3 - 6t$. Find the slope of the tangent line when $t = 0$. HINT: We can say that $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

14. Find h' in terms of f, g, f' and g', if: $h(x) = \dfrac{f(x)g(x)}{f(x) + g(x)}$

15. The volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.

    (a) Find the rate of change of the volume with respect to the radius if the height is constant.

    (b) Find the rate of change of the volume with respect to time if both the height and the radius are functions of time.

16. Find the coordinates of the point on the curve $y = (x - 2)^2$ at which the tangent line is perpendicular to the line $2x - y + 2 = 0$.

17. For what value(s) of A, B, C does the polynomial $y = Ax^2 + Bx + C$ satisfy the differential equation:
$$y'' + y' - 2y = x^2$$
Hint: If $c_1x^2 + c_2x + c_3 = x^2$ for all x, then $c_1 = 1, c_2 = 0, c_3 = 0$.

18. If $V = \sin(w)$, $w = \sqrt{u}$, $u = t^2 + 3t$, compute: The rate of change of V with respect to w, the rate of change of V with respect to u, and the rate of change of V with respect to t.

19. Find all value(s) of k so that $y = \mathrm{e}^{kt}$ satisfies the differential equation:
$$y'' - y' - 2y = 0$$

20. Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

21. Differentiate. You may assume that y is a function of x, if not already defined explicitly. If you use implicit differentiation, solve for $\frac{dy}{dx}$, otherwise you may leave your answer unsimplified.

(a) $y = \log_3(\sqrt{x} + 1)$

(b) $\sqrt{2xy} + xy^3 = 5$

(c) $y = \sqrt{x^2 + \sin(x)}$

(d) $y = \mathrm{e}^{\cos(x)} + \sin(5^x)$

(e) $y = \cot(3x^2 + 5)$

(f) $y = x^{\cos(x)}$

(g) $y = \sqrt{\sin(\sqrt{x})}$

(h) $\sqrt{x} + \sqrt[3]{y} = 1$

(i) $x\tan(y) = y - 1$

(j) $y = \sqrt{x}\,\mathrm{e}^{x^2}(x^2+1)^{10}$ (Hint: Logarithmic Diff)

(k) $y = \sin^{-1}(\tan^{-1}(x))$

(l) $y = \ln|\csc(3x) + \cot(3x)|$

(m) $y = \frac{-2}{\sqrt[4]{t^3}}$

(n) $y = x3^{-1/x}$

(o) $y = x\tan^{-1}(\sqrt{x})$

(p) $y = \mathrm{e}^{2^{\mathrm{e}^x}}$

(q) Let a be a positive constant. $y = x^a + a^x$

(r) $x^y = y^x$

(s) $y = \ln\left(\sqrt{\frac{3x+2}{3x-2}}\right)$

22. Related Rates Extra Practice:

(a) The top of a 25-foot ladder, leaning against a vertical wall, is slipping down the wall at a rate of 1 foot per second. How fast is the bottom of the ladder slipping along the ground when the bottom of the ladder is 7 feet away from the base of the wall?

(b) A 5-foot girl is walking toward a 20-foot lamppost at a rate of 6 feet per second. How fast is the tip of her shadow (cast by the lamppost) moving?

(c) Under the same conditions as above, how fast is the length of the girl's shadow changing?

(d) A rocket is shot vertically upward with an initial velocity of 400 feet per second. Its height s after t seconds is $s = 400t - 16t^2$. How fast is the distance changing from the rocket to an observer on the ground 1800 feet away from the launch site, when the rocket is still rising and is 2400 feet above the ground?

(e) A small funnel in the shape of a cone is being emptied of fluid at the rate of 12 cubic centimeters per second (the tip of the cone is downward). The height of the cone is 20 cm and the radius of the top is 4 cm. How fast is the fluid level dropping when the level stands 5 cm above the vertex of the cone [The volume of a cone is $V = \frac{1}{3}\pi r^2 h$].

(f) A balloon is being inflated by a pump at the rate of 2 cubic inches per second. How fast is the diameter changing when the radius is $\frac{1}{2}$ inch? ($V = \frac{4}{3}\pi r^3$)

(g) A rectangular trough is 8 feet long, 2 feet across the top, and 4 feet deep. If water flows in at a rate of 2 cubic feet per minute, how fast is the surface rising when the water is 1 foot deep?

(h) If a mothball (sphere) evaporates at a rate proportional to its surface area $4\pi r^2$, show that its radius decreases at a constant rate.

(i) If an object is moving along the curve $y = x^3$, at what point(s) is the $y-$coordinate changing 3 times more rapidly than the $x-$coordinate?

23. If f is the function whose graph is given below, let $h(x) = f(f(x))$, and use the graph below to estimate $h'(2)$.