Exam 2 Review Solutions

1. Short Answer:

   (a) How do we define the inverse sine function? (Pay attention to the domain, range and whether the domain, range are angle measures or the ratios of a triangle).
   SOLUTION: We first consider the domain and range of $y = \sin(x)$. Since this function is not 1-1 on its entire domain, we restrict the domain so that the restricted function is 1-1 (meaning the function passes the "horizontal line test"). The standard restriction for the domain is below, together with the range:
   $$-\pi/2 \le x \le \pi/2 \qquad -1 \le y \le 1$$
   Now, the inverse sine is defined to be the inverse to the sine function, and its domain and range are:
   $$-1 \le x \le 1 \qquad -\pi/2 \le y \le \pi/2$$
   Extra: If we ask you to compute $\sin^{-1}(1/\sqrt{2})$, this is the same as: Find angle θ so that $\sin(\theta) = 1/\sqrt{2}$- However, θ must be only between $-\pi/2$ and $\pi/2$ (in this case, $\pi/4$). You might compare this to the answer of something like: Find θ so that $\sin(\theta) = 1/\sqrt{2}$, and $\theta \in [0, 2\pi]$. In this case, you would also add the angle $3\pi/4$.

   (b) What is a *normal line*?
   SOLUTION: The normal line to a function at a given point is the line that is perpendicular to the tangent line (at the same point). Therefore, the slope of the normal line is the negative reciprocal of the slope of the tangent line.

   (c) How do we differentiate a function that involves the absolute value?
   SOLUTION: Rewrite the function into a piecewise form. For example,
   $$|x^2-4| = \begin{cases} x^2-4 & \text{if } x < -2 \text{ or } x > 2 \\ -(x^2-4) & \text{if } -2 \le x \le 2 \end{cases}$$

   (d) $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$

   (e) $\lim_{\theta\to 0} \frac{\cos(\theta)-1}{\theta} = 0$

   (f) $\lim_{\theta\to 0} \frac{\tan(3t)}{\sin(2t)}$ =? (**TYPO:** θ should be t in the limit).
   SOLUTION: We want to manipulate the expression so that we can use the two limits in (d) and (e):
   $$\lim_{t\to 0} \frac{\sin(3t)}{\cos(3t)} \frac{1}{\sin(2t)} = \lim_{t\to 0} \frac{\sin(3t)}{1} \cdot \frac{1}{\cos(3t)} \cdot \frac{1}{\sin(2t)} =$$
   $$\lim_{t\to 0} \frac{3t}{2t} \cdot \frac{\sin(3t)}{3t} \cdot \frac{1}{\cos(3t)} \cdot \frac{2t}{\sin(2t)} = \frac{3}{2} \cdot 1 \cdot 1 \cdot 1 = \frac{3}{2}$$

2. Prove the Reciprocal Rule using the Product Rule (Hint: Start with $f(x) = 1/g(x)$, then write $f(x)g(x) = 1$).
   SOLUTION:
   $$f(x)g(x) = 1 \quad \Rightarrow] \quad f'(x)g(x) + f(x)g'(x) = 0 \quad \Rightarrow \quad f'(x)g(x) = -f(x)g'(x) \quad \Rightarrow \quad f'(x) = -\frac{f(x)}{g'(x)} g(x)$$
   Now, substitute $f(x) = 1/g(x)$ to get the final result:
   $$f'(x) = -\frac{g'(x)}{(g(x))^2}$$

3. Prove the Quotient Rule using the Product and Reciprocal Rules:

$$\left(\frac{f(x)}{g(x)}\right)' = \left(f(x)\frac{1}{g(x)}\right)' = f'(x)\frac{1}{g(x)} + f(x)\left(\frac{-g'(x)}{(g(x))^2}\right)$$

Now get a common denominator ($(g(x))^2$) and simplify a bit:

$$\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

4. True or False, and explain:

(a) The derivative of a polynomial is a polynomial.
True. A polynomial is a function of the form $a_0 + a_1x + a_2x^2 + \ldots a_nx^n$, and its derivative will also have integer powers of x by the Power Rule, nx^{n-1}.

(b) If f is differentiable, then $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$

True:
$$\frac{d}{dx}\sqrt{f(x)} = \frac{d}{dx}(f(x))^{1/2} = \frac{1}{2}(f(x))^{-1/2}f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$$

(c) The derivative of $y = \sec^{-1}(x)$ is the derivative of $y = \cos(x)$.
False. The notation, $\sec^{-1}(x)$ is for the inverse secant function, which is not the reciprocal of the secant.
For extra practice, to get the formula for the derivative of $y = \sec^{-1}(x)$:

$$\sec(y) = x$$

From this, draw a right triangle with one acute angle labelled y, the hypotenuse x and the adjacent length 1. This gives the length of the side opposite: $\sqrt{x^2-1}$. Now differentiate:

$$\sec(y)\tan(y)\frac{dy}{dx} = 1$$

From the triangle, $\sec(y) = x$ and $\tan(y) = \sqrt{x^2-1}$, so:

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$$

(d) $\frac{d}{dx}(10^x) = x10^{x-1}$
False. The Power Rule can only be used for x^n, not a^x. The derivative is $10^x \ln(10)$.

(e) If $y = \ln|x|$, then $y' = \frac{1}{x}$.
TRUE. To see this, re-write the function:

$$y = \begin{cases} \ln(x) & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases} \quad \Rightarrow \quad y' = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{-x}\cdot(-1) & \text{if } x < 0 \end{cases}$$

From which we see that $y' = \frac{1}{x}$, $x \neq 0$.

(f) The equation of the tangent line to $y = x^2$ at $(1, 1)$ is:

$$y - 1 = 2x(x-1)$$

False. This is the equation of a parabola, not a line. The derivative, $y' = 2x$ gives a formula for the slope of the tangent line, and is not the slope itself. To get the slope, we evaluate the derivative at $x = 1$, which gives $y' = 2$. The slope of the tangent line is therefore $y - 1 = 2(x - 1)$.

(g) If $y = e^2$, then $y' = 2e$
False. e^2 is a constant, so the derivative is zero.

(h) If $y = |x^2 - x|$, then $y' = |2x - 1|$.
False. First, rewrite y, then differentiate:

$$y = \begin{cases} x^2 - x & \text{if } x \leq 0 \text{ or } x \geq 1 \\ -(x^2 - x) & \text{if } 0 < x < 1 \end{cases} \quad \Rightarrow \quad y' = \begin{cases} 2x - 1 & \text{if } x < 0 \text{ or } x > 1 \\ -(2x - 1) & \text{if } 0 < x < 1 \end{cases}$$

Compare this to $|2x - 1|$:

$$|2x - 1| = \begin{cases} 2x - 1 & \text{if } x \geq 1/2 \\ -(2x - 1) & \text{if } x < 1/2 \end{cases}$$

By the way, we also note that y is NOT differentiable at $x = 0$ or at $x = 1$ by checking to see what the derivatives are approaching as $x \to 0$ and as $x \to 1$.

(i) If $y = ax + b$, then $\frac{dy}{da} = x$
True. $\frac{dy}{da}$ means that we treat a as an independent variable, and x, b as constants.

5. Find the equation of the tangent line to $x^3 + y^3 = 3xy$ at the point $(\frac{3}{2}, \frac{3}{2})$.

We need to find the slope, $\left.\frac{dy}{dx}\right|_{x=3/2, y=3/2}$

$$3x^2 + 3y^2 y' = 3y + 3xy' \Rightarrow y'(3y^2 - 3x) = 3y - 3x^2 \Rightarrow y' = \frac{y - x^2}{y^2 - x}$$

Substituting $x = 3/2, y = 3/2$ gives $y' = -1$, so the equation of the tangent line is $y - 3/2 = -1(x - 3/2)$

6. If $f(0) = 0$, and $f'(0) = 2$, find the derivative of $f(f(f(f(x))))$ at $x = 0$.

A cute chain rule problem! Here we go- the derivative is:

$$f'(f(f(f(x)))) \cdot f'(f(f(x))) \cdot f'(f(x)) \cdot f'(x)$$

Now subsitute $x = 0$ and evaluate:

$$f'(f(f(f(0)))) \cdot f'(f(f(0))) \cdot f'(f(0)) \cdot f'(0) =$$

$$f'(0) \cdot f'(0) \cdot f'(0) \cdot f'(0) = 2^4 = 16$$

7. If $f(x) = 2x + \mathrm{e}^x$, find the equation of the tangent line to the INVERSE of f at $(1, 0)$.

First, we verify that $(0, 1)$ is on the graph of f:

$$f(0) = 2 \cdot 0 + \mathrm{e}^0 = 1$$

We know that, if $f'(0) = m$, then $\left.\frac{df^{-1}}{dx}\right|_{x=1} = \frac{1}{m}$.

Now, $f'(x) = 2 + \mathrm{e}^x$, so $f'(0) = 3$. Therefore, the slope of the tangent line to the inverse of f at $x = 1$ is $\frac{1}{3}$, and the equation is then:

$$y - 0 = \frac{1}{3}(x - 1) \text{ or } y = \frac{1}{3}x - \frac{1}{3}$$

8. Derive the formula for the derivative of $y = \csc^{-1}(x)$ using implicit differentiation:

   SOLUTION: $\csc(y) = x$, so that we can differentiate both sides:

$$-\csc(y)\cot(y)y' = 1 \quad \Rightarrow \quad y' = \frac{-1}{\csc(y)\cot(y)}$$

   Now, to simplify (and get a formula in terms of x), draw a triangle satisfying $\csc(y) = x$- That is, label an angle y, then the length of the hypotenuse is x and the length of the side adjacent is 1. The length of the side opposite is therefore $\sqrt{x^2-1}$ (by the Pythagorean Theorem), and

$$y' = \frac{-1}{\csc(y)\cot(y)} = \frac{-1}{x\sqrt{x^2-1}}$$

9. Find the equation of the tangent line to $\sqrt{y} + xy^2 = 5$ at the point $(4, 1)$.

   Implicit differentiation gives:

$$\frac{1}{2}y^{-1/2}y' + y^2 + 2xyy' = 0$$

   Now we could solve for y' now, or substitute $x = 4, y = 1$:

$$\frac{1}{2}1^{-1/2}y' + 1^2 + 2(4)(1)y' = 0 \Rightarrow y' = -2/17$$

   The equation of the tangent line is $y - 1 = \frac{-2}{17}(x - 4)$

10. If $s^2t + t^3 = 1$, find $\frac{dt}{ds}$ and $\frac{ds}{dt}$.

    The notation $\frac{dt}{ds}$ means that we are treating t as a function of s. Therefore, we have:

$$2st + s^2\frac{dt}{ds} + 3t^2\frac{dt}{ds} = 0 \Rightarrow \frac{dt}{ds} = \frac{-2st}{s^2+3t^2}$$

    For the second part, we have two choices. One choice is to treat s as a function of t and differentiate:

$$2s\frac{ds}{dt}t + s^2 + 3t^2 = 0 \Rightarrow \frac{ds}{dt} = \frac{-(s^2+3t^2)}{2st}$$

    Another method is to realize that:

$$\frac{ds}{dt} = \frac{1}{\frac{dt}{ds}} = \frac{1}{\frac{-2st}{s^2+3t^2}} = -\frac{s^2+3t^2}{2st}$$

    Cool!

11. If $y = x^3 - 2$ and $x = 3z^2 + 5$, then find $\frac{dy}{dz}$.

    We see that $\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz}$, so we calculate $\frac{dy}{dx}$ and $\frac{dx}{dz}$:

$$\frac{dy}{dx} = 3x^2, \quad \frac{dx}{dz} = 6z$$

    so that

$$\frac{dy}{dz} = 3x^2 \cdot 6z = 3(3z^2+5)^2 \cdot 6z = 18z(3z^2+5)^2$$

12. A space traveler is moving from left to right along the curve $y = x^2$. When she shuts off the engines, she will go off along the tangent line at that point. At what point should she shut off the engines in order to reach the point $(4, 15)$?

The unknown in the problem is a point on the parabola $y = x^2$. Let's label that point as (a, a^2). Now our goal is to find a.

First, the line will go through both (a, a^2) and $(4, 15)$, so the slope will satisfy:

$$m = \frac{a^2 - 15}{a - 4}$$

Secondly, the line will be a tangent line, so the slope will also be $m = 2a$. Equating these, we can solve for a:

$$\frac{a^2 - 15}{a - 4} = 2a \Rightarrow a^2 - 15 = 2a(a - 4) \Rightarrow a^2 - 8a + 15 = 0 \Rightarrow a = 3, a = 5$$

Since we're moving from left to right, we would choose the smaller of these, $a = 3$.

13. A particle moves in the plane according to the law $x = t^2 + 2t$, $y = 2t^3 - 6t$. Find the slope of the tangent line when $t = 0$.

The slope is $\frac{dy}{dx}$, but we can only compute $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Note however, that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t^2 - 6}{2t + 2}$$

So, at $t = 0$, $\frac{dy}{dx} = -3$.

14. Find h' in terms of f' and g', if: $h(x) = \dfrac{f(x)g(x)}{f(x) + g(x)}$

SOLUTION:

$$h'(x) = \frac{(f'g + fg')(f + g) - (fg)(f' + g')}{(f + g)^2} = \frac{f^2g' + f'g^2}{(f + g)^2}$$

15. The volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.

(a) Find the rate of change of the volume with respect to the radius if the height is constant.

SOLUTION: The question is asking for dV/dr, when h is constant:

$$\frac{dV}{dr} = \frac{2}{3}\pi r h$$

(b) Find the rate of change of the volume with respect to time if both the height and the radius are functions of time.

SOLUTION: Now we want dV/dt if h, r are functions of time. That means the formula for V looks like:

$$V(t) = \frac{1}{3}\pi (r(t))^2 h(t)$$

So we'll use the product rule (and factor the constant out):

$$\frac{dV}{dt} = \frac{\pi}{3}\left(2r(t)r'(t)h(t) + (r(t))^2 h'(t)\right)$$

16. Find the coordinates of the point on the curve $y = (x-2)^2$ at which the tangent line is perpendicular to the line $2x - y + 2 = 0$.

First, recall that two slopes are perpendicular if they are negative reciprocals (like $-3, \frac{1}{3}$).

The slope of the given line is 2, so we want a slope of $-\frac{1}{2}$.

The x that will provide this slope is found by differentiating:

$$y' = 2(x-2) \Rightarrow 2(x-2) = -\frac{1}{2} \Rightarrow x = \frac{7}{4} \text{ from which } y = \frac{1}{16}$$

17. For what value(s) of A, B, C does the polynomial $y = Ax^2 + Bx + C$ satisfy the differential equation:

$$y'' + y' - 2y = x^2$$

As we did in class, compute the derivatives of y and substitute into the equation:

$$y' = 2Ax + B, \quad y'' = 2A$$

so that:

$$2A + 2Ax + B - 2(Ax^2 + Bx + C) = x^2$$

Now collect coefficients to get:

$$(-2A)x^2 + (2A - 2B)x + (B + 2A - 2C) = x^2 \text{ for all } x$$

so the coefficient of x^2 on both sides of the equation must be the same,

$$-2A = 1$$

The coefficient of x on both sides of the equation must be the same,

$$2A - 2B = 0$$

And the constant terms must be the same,

$$B + 2A - 2C = 0$$

This gives the solution, $A = -\frac{1}{2}$, $B = -\frac{1}{2}, C = -\frac{3}{4}$.

18. If $V = \sin(w)$, $w = \sqrt{u}$, $u = t^2 + 3t$, compute: The rate of change of V with respect to w, the rate of change of V with respect to u, and the rate of change of V with respect to t.

- $\dfrac{dV}{dw} = \cos(w)$
- $\dfrac{dV}{du} = \dfrac{dV}{dw} \cdot \dfrac{dw}{du} = \cos(w) \cdot \dfrac{1}{2\sqrt{u}} = \dfrac{\cos(\sqrt{u})}{2\sqrt{u}}$
- $\dfrac{dV}{dt} = \dfrac{dV}{dw} \cdot \dfrac{dw}{du} \cdot \dfrac{du}{dt} = \cos(w) \cdot \dfrac{1}{2\sqrt{u}} \cdot (2t+3) = \dfrac{\cos(\sqrt{t^2+3t})}{2\sqrt{t^2+3t}} \cdot (2t+3)$

19. Find all value(s) of k so that $y = \mathrm{e}^{kt}$ satisfies the differential equation: $y'' - y' - 2y = 0$.

First, differentiate y, then substitute:

$$y = \mathrm{e}^{kt} \quad \Rightarrow \quad y' = k\mathrm{e}^{kt} \quad \Rightarrow \quad y'' = k^2\mathrm{e}^{kt}$$

so that:

$$k^2\mathrm{e}^{kt} - k\mathrm{e}^{kt} - 2\mathrm{e}^{kt} = 0 \Rightarrow \mathrm{e}^{kt}(k^2 - k - 2) = 0$$

Since $\mathrm{e}^{kt} = 0$ has no solution, the only solution(s) come from:

$$k^2 - k - 2 = 0 \quad \Rightarrow \quad (k+1)(k-2) = 0$$

so $k = -1$, $k = 2$ are the two values of k.

20. Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

Implicit differentiation gives:

$$2x + 4yy' = 0 \Rightarrow y' = -\frac{x}{2y}$$

To have a slope of 1 will mean that: $\frac{-x}{2y} = 1$, so that $x = -2y$.

This means that any point on the ellipse satisfying $x = -2y$ will have a tangent line with slope 1, so we look for the points of intersection between $x = -2y$ and the ellipse $x^2 + 2y^2 = 1$. Therefore, we substitute for x and get:

$$(-2y)^2 + 2y^2 = 1 \quad \Rightarrow \quad 6y^2 = 1 \quad \Rightarrow \quad y = \pm\frac{1}{\sqrt{6}}$$

To get the $x-$ coordinate, you can either backsubstitute into the equation for the ellipse:

$$x^2 + 2\left(\frac{\pm 1}{\sqrt{6}}\right)^2 = 1 \quad \Rightarrow \quad x = \pm\sqrt{\frac{2}{3}}$$

Or you can use the fact that $x = -2y$, so in that case $x = \mp 2/\sqrt{6}$, or $(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}})$ or $(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}})$

NOTE: These look like different values of x, but either is fine since:

$$\frac{2}{\sqrt{6}} = \frac{2}{\sqrt{2}\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}$$

21. Differentiate. You may assume that y is a function of x, if not already defined explicitly.

(a) $y = \log_3(\sqrt{x} + 1)$ Use the Chain Rule:

$$y' = \frac{1}{(\sqrt{x} + 1)\ln(3)} \cdot \frac{1}{2\sqrt{x}}$$

(b) $\sqrt{2xy} + xy^3 = 5$ (and solve for $\frac{dy}{dx}$)
Before solving for y', we get:

$$\frac{1}{2}(2xy)^{-1/2}(2y + 2xy') + y^3 + 3xy^2y' = 0$$

$$y' = \frac{-(y(2xy)^{-1/2} + y^3)}{x(2xy)^{-1/2} + 3xy^2}$$

(c) $y = \sqrt{x^2 + \sin(x)}$

$$\frac{dy}{dx} = \frac{1}{2}\left(x^2 + \sin(x)\right)^{-1/2}(2x + \cos(x))$$

(d) $y = \mathrm{e}^{\cos(x)} + \sin(5^x)$

$$y' = \mathrm{e}^{\cos(x)}(-\sin(x)) + \cos(5^x) \cdot 5^x \ln(5)$$

(e) $y = \cot(3x^2 + 5)$

$$y' = -\csc^2(3x^2 + 5)(6x) = -6x\csc^2(3x^2 + 5)$$

(f) $y = x^{\cos(x)}$

Use logarithmic differentiation: $\ln(y) = \cos(x) \cdot \ln(x)$, so that

$$\frac{1}{y}y' = -\sin(x)\ln(x) + \cos(x) \cdot \frac{1}{x}$$

Multiply both sides of the equation by y, and back substitute $y = x^{\cos(x)}$ to get:

$$y' = x^{\cos(x)}\left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right)$$

(g) $y = \sqrt{\sin(\sqrt{x})}$

$$y' = \frac{1}{2}(\sin(x^{1/2}))^{-1/2}\cos(x^{1/2})\frac{1}{2}x^{-1/2}$$

(h) $\sqrt{x} + \sqrt[3]{y} = 1$

$$\frac{1}{2}x^{-1/2} + \frac{1}{3}y^{-2/3}y' = 0$$

$$y' = -\frac{3y^{2/3}}{2x^{1/2}}$$

(i) $x\tan(y) = y - 1$

$$\tan(y) + x\sec^2(y)y' = y'$$

$$\frac{\tan(y)}{1 - x\sec^2(y)} = y'$$

(j) $y = \sqrt{x}\,\mathrm{e}^{x^2}(x^2+1)^{10}$ (Hint: Logarithmic Diff)

First, we rewrite so that:

$$\ln(y) = \ln(\sqrt{x}\,\mathrm{e}^{x^2}(x^2+1)^{10})$$

Use the rules of logarithms to re-write this as the sum:

$$\ln(y) = \frac{1}{2}\ln(x) + x^2\ln(e) + 10\ln(x^2+1) = \frac{1}{2}\ln(x) + x^2 + 10\ln(x^2+1)$$

So far, we've only done algebra. Now it's time to differentiate:

$$\frac{1}{y}y' = \frac{1}{2}\cdot\frac{1}{x} + 2x + 10\cdot\frac{1}{x^2+1}\cdot 2x$$

Simplifying, multiplying through by y:

$$y' = y\left(\frac{1}{2x} + 2x + \frac{20x}{x^2+1}\right)$$

Finally, back substitute y:

$$y' = \sqrt{x}\,\mathrm{e}^{x^2}(x^2+1)^{10}\cdot\left(\frac{1}{2x} + 2x + \frac{20x}{x^2+1}\right)$$

(k) $y = \sin^{-1}\left(\tan^{-1}(x)\right)$

This is a composition, so use the chain rule:

$$y' = \frac{1}{\sqrt{1 - \left(\tan^{-1}(x)\right)^2}}\cdot\frac{1}{x^2+1}$$

(l) $y = \ln|\csc(3x) + \cot(3x)|$
Recall that the derivative of $\ln|x|$ is $\frac{1}{x}$, so using the Chain Rule:

$$y' = \frac{1}{\csc(3x) + \cot(3x)} \cdot \left[-\csc(3x)\cot(3x) \cdot 3 - \csc^2(3x) \cdot 3\right]$$

Which can be simplified:

$$y' = \frac{-3\csc(3x)(\cot(3x) + \csc(3x))}{\csc(3x) + \cot(3x)} = -3\csc(3x)$$

(m) $y = \frac{-2}{\sqrt[4]{t^3}}$ First, note that $y = -2t^{-3/4}$ so $y' = \frac{3}{2}t^{-7/4}$

(n) $y = x3^{-1/x}$

$$y' = 3^{-1/x} + x3^{-1/x}\ln(3) \cdot x^{-2}$$

(o) $y = x\tan^{-1}(\sqrt{x})$
Overall, use the product rule (then a chain rule):

$$y' = \tan^{-1}(\sqrt{x}) + x \cdot \left(\frac{1}{(\sqrt{x})^2 + 1} \cdot \frac{1}{2\sqrt{x}}\right) == \tan^{-1}(\sqrt{x}) + \frac{\sqrt{x}}{2(x^2+1)}$$

(p) $y = e^{2^{e^x}}$ Before putting in the values, note that this derivative will be in the form:

$$y' = e^{(\,)} \cdot \frac{d}{dx}(2^{(\,)}) \cdot \frac{d}{dx}e^x = e^{(\,)} \cdot 2^{(\,)}\ln(2) \cdot e^x$$

Putting in the appropriate expressions gives us:

$$y' = e^{2^{e^x}} 2^{e^x} \ln(2)e^x$$

(q) Let a be a positive constant. $y = x^a + a^x$

$$y' = ax^{a-1} + a^x\ln(a)$$

(r) $x^y = y^x$
We must use logs first, since the exponents have x and y:

$$\ln(x^y) = \ln(y^x) \Rightarrow y\ln(x) = x\ln(y) \Rightarrow y'\ln(x) + y \cdot \frac{1}{x} = \ln(y) + x \cdot \frac{1}{y}y'$$

Now isolate and solve for y':

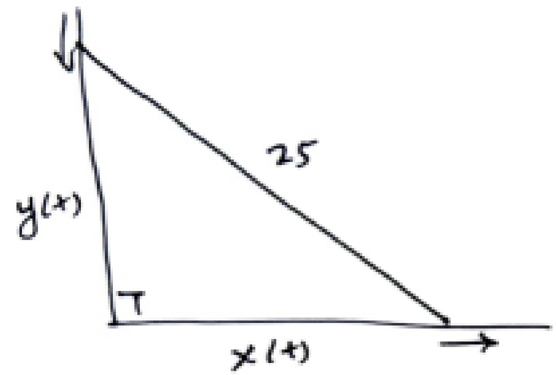
$$y'\left(\ln(x) - \frac{x}{y}\right) = \ln(y) - \frac{y}{x} \Rightarrow y' = \frac{\ln(y) - \frac{y}{x}}{\ln(x) - \frac{x}{y}} = \frac{y(x\ln(y) - y)}{x(y\ln(x) - x)}$$

(s) Rewrite first: $y = \ln\left(\sqrt{\frac{3x+2}{3x-2}}\right) = \frac{1}{2}\left(\ln(3x+2) - \ln(3x-2)\right)$ Now y' can be computed:

$$y' = \frac{1}{2}\left(\frac{3}{3x+2} - \frac{3}{3x-2}\right) = \frac{1}{2}\left(\frac{3(3x-2) - 3(3x+2)}{(3x+2)(3x-2)}\right) = \frac{-6}{(3x+2)(3x-2)}$$

22. Related Rates Extra Practice

(a) The top of a 25-foot ladder, leaning against a vertical wall, is slipping down the wall at a rate of 1 foot per second. How fast is the bottom of the ladder slipping along the ground when the bottom of the ladder is 7 feet away from the base of the wall?



First, make a sketch of a triangle whose hypotenuse is the ladder. Let $y(t)$ be the height of the ladder with the vertical wall, and let $x(t)$ be the length of the bottom of the ladder with the vertical wall. Then

$$x^2(t) + y^2(t) = 25^2$$

The problem can then be interpreted as: If $\frac{dy}{dt} = -1$, what is $\frac{dx}{dt}$ when $x = 7$?

Differentiating with respect to time:

$$2x(t)\frac{dx}{dt} + 2y(t)\frac{dy}{dt} = 0$$

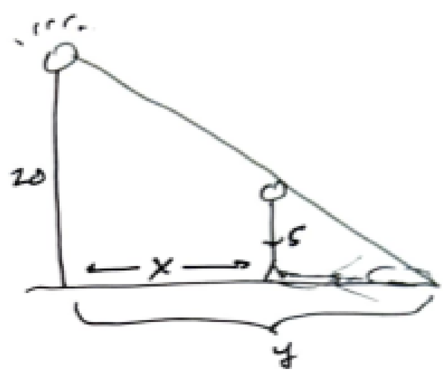
we have numbers for $\frac{dy}{dt}$ and $x(t)$- we need a number for y in order to solve for $\frac{dx}{dt}$. Use the original equation, and

$$7^2 + y^2(t) = 25^2 \Rightarrow y = 24$$

(NOTE: If you had no calculator, you could leave $y = \sqrt{25^2 - 7^2}$)
Now plug everything in and solve for $\frac{dx}{dt}$:

$$2 \cdot 7 \cdot \frac{dx}{dt} + 2 \cdot 24 \cdot (-1) = 0 \Rightarrow \frac{dx}{dt} = \frac{24}{7}$$

(b) A 5-foot girl is walking toward a 20-foot lamppost at a rate of 6 feet per second. How fast is the tip of her shadow (cast by the lamppost) moving?



Let $x(t)$ be the distance of the girl to the base of the post, and let $y(t)$ be the distance of the tip of the shadow to the base of the post. If you've drawn the right setup, you should see similar triangles...

$$\frac{\text{Hgt of post}}{\text{Hgt of girl}} = \frac{\text{Dist of tip of shadow to base}}{\text{Dist of girl to base}}$$

In our setup, this means:

$$\frac{20}{5} = \frac{y(t)}{y(t) - x(t)}$$

With a little simplification, we get:

$$3y(t) = 4x(t)$$

We can now interpret the question as asking what $\frac{dy}{dt}$ is when $\frac{dx}{dt} = -6$. Differentiating, we get

$$3\frac{dy}{dt} = 4\frac{dx}{dt}$$

so that the final answer is $\frac{dy}{dt} = -8$, which we interpret to mean that the tip of the shadow is approaching the post at a rate of 8 feet per second.

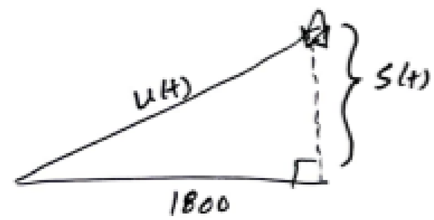
(c) Under the same conditions as above, how fast is the length of the girl's shadow changing?
Let $L(t)$ be the length of the shadow at time t. Then, by our previous setup,

$$L(t) = y(t) - x(t)$$

so $\frac{dL}{dt} = \frac{dy}{dt} - \frac{dx}{dt} = -8 - (-6) = -2$.

(d) A rocket is shot vertically upward with an initial velocity of 400 feet per second. Its height s after t seconds is $s = 400t - 16t^2$. How fast is the distance changing from the rocket to an observer on the ground 1800 feet away from the launch site, when the rocket is still rising and is 2400 feet above the ground?

We can form a right triangle, where the launch site is the vertex for the right angle. The height is $s(t)$, given in the problem, the length of the second leg is fixed at 1800 feet. Let $u(t)$ be the length of the hypotenuse. Now we have:



$$u^2(t) = s^2(t) + 1800^2$$

and we can interpret the question as asking what $\frac{du}{dt}$ is when $s(t) = 2400$. Differentiating, we get

$$2u(t)\frac{du}{dt} = 2s(t)\frac{ds}{dt} \text{ or } u(t)\frac{du}{dt} = s(t)\frac{ds}{dt}$$

To solve for $\frac{du}{dt}$, we need to know $s(t)$, $u(t)$ and $\frac{ds}{dt}$. We are given $s(t) = 2400$, so we can get $u(t)$:

$$u(t) = \sqrt{2400^2 - 1800^2} = 3000$$

Now we need $\frac{ds}{dt}$. We are given that $s(t) = 400t - 16t^2$, so $\frac{ds}{dt} = 400 - 32t$. That means we need t. From the equation for $s(t)$,

$$2400 = 400t - 16t^2 \Rightarrow -16t^2 + 400t - 2400 = 0$$

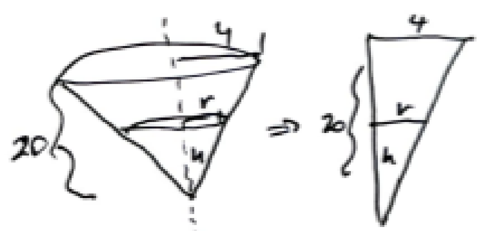
Solve this to get $t = 10$ or $t = 15$. Our rocket is on the way up, so we choose $t = 10$. Finally we can compute $\frac{ds}{dt} = 400 - 32(10) = 80$. Now,

$$3000\frac{du}{dt} = 2400(80)$$

so $\frac{du}{dt} = 64$ feet per second (at time 10).

(e) A small funnel in the shape of a cone is being emptied of fluid at the rate of 12 cubic centimeters per second (the tip of the cone is downward). The height of the cone is 20 cm and the radius of the top is 4 cm. How fast is the fluid level dropping when the level stands 5 cm above the vertex of the cone [The volume of a cone is $V = \frac{1}{3}\pi r^2 h$].

Draw a picture of an inverted cone. The radius at the top is 4, and the overall height is 20. Inside the cone, draw some water at a height of $h(t)$, with radius $r(t)$. We are given information about the rate of change of volume of water, so we are given that $\frac{dV}{dt} = -12$. Note that the formula for volume is given in terms of r and h, but we only want $\frac{dh}{dt}$. We need a relationship between r and h...



You should see similar triangles (Draw a line right through the center of the cone. This, and the line forming the top radius are the two legs. The outer edge of the cone forms the hypotenuse).

$$\frac{\text{radius of top}}{\text{radius of water level}} = \frac{\text{overall height}}{\text{height of water}} \Rightarrow \frac{4}{r} = \frac{20}{h}$$

so that $r = \frac{h}{5}$. Substituting this into the formula for the volume will give the volume in terms of h alone:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\frac{h}{5})^2 h = \frac{1}{75}\pi h^3$$

Now,

$$\frac{dV}{dt} = \frac{3\pi}{75} h^2 \frac{dh}{dt}$$

and we know $\frac{dV}{dt} = -12$, $h = 5$, so

$$\frac{dh}{dt} = \frac{-12}{\pi}$$

(f) A balloon is being inflated by a pump at the rate of 2 cubic inches per second. How fast is the diameter changing when the radius is $\frac{1}{2}$ inch?
The volume is $V = \frac{4}{3}\pi r^3$ (this formula would be given to you on an exam/quiz). If we let h be the diameter, then we know that $2r = h$, so we can make V depend on diameter instead of radius:

$$V = \frac{4}{3}\pi(\frac{h}{2})^3 = \frac{\pi}{6}h^3$$

Now, the question is asking for $\frac{dh}{dt}$ when $h = 1$, and we are given that $\frac{dV}{dt} = 2$. Differentiate, and

$$\frac{dV}{dt} = \frac{\pi}{6}3h^2\frac{dh}{dt} = \frac{\pi}{2}h^2\frac{dh}{dt}$$

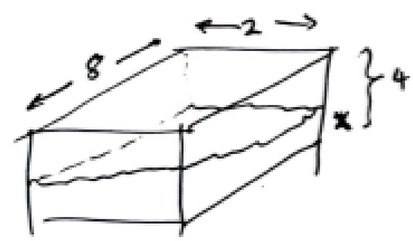
so that $\frac{dh}{dt} = \frac{4}{\pi}$.

(g) A rectangular trough is 8 feet long, 2 feet across the top, and 4 feet deep. If water flows in at a rate of 2 cubic feet per minute, how fast is the surface rising when the water is 1 foot deep?

The trough is a rectangular box. Let $x(t)$ be the height of the water at time t. Then the volume of the water is:

$$V = 16x \Rightarrow \frac{dV}{dt} = 16\frac{dx}{dt}$$

Put in $\frac{dV}{dt} = 2$ to get that $\frac{dx}{dt} = \frac{1}{8}$.



(h) If a mothball (sphere) evaporates at a rate proportional to its surface area $4\pi r^2$, show that its radius decreases at a constant rate.
Let $V(t)$ be the volume at time t. We are told that

$$\frac{dV}{dt} = kA(t) = k4\pi r^2$$

We want to show that $\frac{dr}{dt}$ is constant.
By the formula for $V(t) = \frac{4}{3}\pi r^3$, we know that:

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

Now compare the two formulas for $\frac{dV}{dt}$, and we see that $\frac{dr}{dt} = k$, which was the constant of proportionality!

(i) If an object is moving along the curve $y = x^3$, at what point(s) is the $y-$coordinate changing 3 times more rapidly than the $x-$coordinate?

Let's differentiate:

$$\frac{dy}{dt} = 3x^2 \frac{dx}{dt}$$

From this, we see that if we want $\frac{dy}{dt} = 3\frac{dx}{dt}$, then we must have $x = \pm 1$. We also could have $x = 0, y = 0$, since 0 is 3 times 0. All the points on the curve are therefore:

$$(0,0), (-1,1), (1,1)$$

23. For the last problem, estimate the derivatives- To me, it looks like the derivative at 1 is less steep than the derivative at 2, although if you guessed that both are fairly small negative numbers, that would be OK. In this case then:

$$h'(2) = f'(f(2))f'(2) = f'(1)f'(2) \approx (-1/2)(-1) = 1$$