Exam 3 Review solutions

1. Short Answer:

   (a) Give the definition of a **critical point** for a function f:
   A critical point is a value of x for which $f'(x) = 0$ or does not exist (from the domain of f).

   (b) State the three "Value Theorems" (don't just name them, but also state each):

   - Intermediate Value Theorem: If f is continuous on $[a, b]$ and N is any number between $f(a)$ and $f(b)$, then there is a c in $[a, b]$ such that $f(c) = N$.
   - Extreme Value Theorem: If f is continuous on $[a, b]$, then f attains an absolute maximum and minimum on $[a, b]$.
   - Mean Value Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a c in (a, b) such that

   $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

   (c) What is the procedure for finding the maximum or minimum of a function $y = f(x)$ on a closed interval, $[a, b]$.
   SOLUTION: This is the closed interval method. Find the critical points (in the interval $[a, b]$. Build a table using the CPs and endpoints. The largest $y-$value is your absolute max, the smallest $y-$value is your absolute min.

   (d) How do we determine if a function has a local maximum or minimum?
   SOLUTION: First we compute the critical points. Then we can use either the first or second derivative test. The first derivative test says that if the sign of the first derivative changes from positive to negative, we have a local maximum. If the sign changes from negative to positive, we have a local minimum. If there is no sign change, we do not have a local extreme point.
   For the second derivative test, if $f'(c) = 0$, we check $f''(c)$. If that value is positive, then c is where a local minimum occurs. If that value is negative, then c is where a local maximum occurs.

   (e) What is meant by *linearizing* a function?
   SOLUTION: We replace f by $L(x)$ close by some base point $x = a$. In this case, we say

   $$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

2. True or False, and give a short reason:

   (a) If $f'(a) = 0$, then there is a local maximum or local minimum at $x = a$.
   FALSE: For example, $f(x) = x^3$ at $x = 0$.

   (b) There is a vertical asymptote at $x = 2$ for $\frac{\sqrt{x^2+5}-3}{x^2-2x}$
   NOTE: I suspect it is FALSE because both the numerator and denominator are zero at $x = 2$. To show it, I need to take the limit:

   $$\lim_{x \to 2} \frac{\sqrt{x^2+5}-3}{x^2-2x} \cdot \frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x(x-2)(\sqrt{x^2+5}+3)} = \frac{1}{3}$$

   so, it is FALSE.

   (c) If f has a global minimum at $x = a$, then $f'(a) = 0$.
   FALSE. It could be that $x = a$ is an endpoint for an interval, or a point where $f'(a)$ does not exist.

(d) If $f''(2) = 0$, then $(2, f(2))$ is an inflection point for f.
FALSE. An inflection point is where the curve changes convexity. If the second derivative is continuous, then the answer would be "true".

In the following, "increasing" or "decreasing" will mean for all real numbers x:

(e) If $f(x)$ is increasing, and $g(x)$ is increasing, then $f(x) + g(x)$ is increasing.
TRUE: Since f, g are increasing, f', g' are both positive so $f' + g'$ is positive as well.

(f) If $f(x)$ is increasing, and $g(x)$ is increasing, then $f(x)g(x)$ is increasing.
FALSE: It might be true, but

$$(fg)' = f'g + fg'$$

so we might have that $g < 0$ and $f < 0$, for example. In that case, the derivative would be negative.

(g) If $f(x)$ is increasing, and $g(x)$ is decreasing, then $f(g(x))$ is decreasing.
TRUE:

$$(f(g(x)))' = f'(g(x))g'(x)$$

so f' (evaluated at $g(x)$) is positive, and g' is positive.

3. Find the global maximum and minimum of the given function on the interval provided:

(a) $f(x) = \sqrt{9 - x^2}$, $[-1, 2]$
SOLUTION: $f'(x) = -x/\sqrt{9 - x^2}$, so we add the critical point $x = 0$. Build a table:

x	-1	0	2
y	2.82	3.0	2.23

So the global minimum is $y = 2.23$ and the global maximum is $y = 3.0$

(b) $g(x) = x - 2\cos(x)$, $[-\pi, \pi]$
SOLUTION: $g'(x) = 1 + 2\sin(x)$. Use a triangle and/or unit circle to find the values of $x = -5\pi/6$ and $x = -\pi/6$. Now the table:

x	$-\pi$	$-5\pi/6$	$-\pi/6$	π
y	-1.14	-0.88	-2.25	5.14

The global minimium is approx -2.25 and the global max is approx 5.14.

4. Find the regions where f is increasing/decreasing: $f(x) = \frac{x}{(1+x)^2}$

SOLUTION: Simplifying the derivative, we get:

$$f'(x) = \frac{1 - x}{(1 + x)^3}$$

Sign chart (include $x = -1$ although it is a vertical asymptote):

$f'(x)$	$-$	$+$	$-$
	$x < -1$	$-1 < x < 1$	$x > 1$

Therefore, f is decreasing if $x < -1$ and if $x > 1$, and increasing if $-1 < x < 1$.

5. For each function below, determine (i) where f is increasing/decreasing, (ii) where f is concave up/concave down, and (iii) find the local extrema.

(a) $f(x) = x^3 - 12x + 2$ (See Exercise 33, 4.3)
Hint: $f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x-2)(x+2)$, then look at a sign chart, and $f''(x) = 6x$.

(b) $f(x) = x\sqrt{6-x}$ (See Exercise 39, 4.3)
Hints: The domain is $x \le 6$, and we can simplify

$$f'(x) = \frac{3}{2} \cdot \frac{x-4}{\sqrt{6-x}}$$

$$f''(x) = \frac{3}{4} \cdot \frac{x-8}{(6-x)^{3/2}}$$

(c) $f(x) = x - \sin(x)$, $0 < x < 4\pi$ (See Exercise 44. 4.3)
SOLUTION: $f'(x) = 1 - \cos(x)$, and $\cos(x) = 1$ at $x = 0, 2\pi, 4\pi$. Since we do not include $0, 4\pi$ in the interval, we have:

$f'(x)$	$+$	$+$
	$0 < x < 2\pi$	$2\pi < x < 4\pi$

Therefore, $f'(x) > 0$, and f is always increasing.
For concavity, $f''(x) = \sin(x)$. The sine is posiive (so f is increasing) on $(0, \pi)$ and $(2\pi, 3\pi)$. The sine is negative (so f is decreasing) on $(\pi, 2\pi)$ and $(3\pi, 4\pi)$.
For local extrema, the only critical point is 2π, and from our analysis of the first derivative, we see that this is a plateau, not a local extreme point (so there are no local extrema).

6. Suppose $f(3) = 2$, $f'(3) = \frac{1}{2}$, and $f'(x) > 0$ and $f''(x) < 0$ for all x.

(a) Sketch a possible graph for f.
SOLUTION: At the point $(3, 2)$, the function increases and is concave down.

(b) How many roots does f have? (Explain):
SOLUTION: The function f can have at most 1 root. If it had two roots, Rolle's theorem would mean that $f'(x) = 0$ for some x between the roots- But we're told that $f'(x) > 0$.

(c) Is it possible that $f'(2) = 1/3$? Why?
SOLUTION: Since $f'(3) = 1/2$ and $1/3 < 1/2$, then this implies that f' is increasing. However, that implies that $f'' > 0$, but $f''(x) < 0$ for all x. Therefore, the given value of $f'(x)$ is not possible.

7. Let $f(x) = 2x + e^x$.

Show that f has exactly one real root.

- We see that $f(-1) = -2 + e^{-1} < 0$ and $f(0) = 1 > 0$. Therefore, by the IVT there is at least one real root.
- The derivative is $f'(x) = 2 + e^x$, which is always positive (since e^x is always greater than 0). Therefore, by Rolle's Theorem, there is at most 1 real root.
- By the previous 2 items, there is exactly one (real) root.

8. Suppose that $1 \le f'(x) \le 3$ for all $0 \le x \le 2$, and $f(0) = 1$. What is the largest and smallest that $f(2)$ can be?

SOLUTION: For the maximum,

$$\frac{f(2) - f(0)}{2 - 0} = f'(c) \le 3 \quad \Rightarrow \quad \frac{f(2) - 1}{2} \le 3 \quad \Rightarrow \quad f(2) \le 6 + 1 = 7$$

Similarly, for the minimum,

$$\frac{f(2)-f(0)}{2-0} = f'(c) \geq 1 \quad \Rightarrow \quad \frac{f(2)-1}{2} \geq 1 \quad \Rightarrow \quad f(2) \leq 2+1 = 3$$

Therefore, $3 \leq f(2) \leq 7$.

9. Linearize at $x = 0$:

$$y = \sqrt{x+1}\mathrm{e}^{-x^2}$$

Use the linearization to estimate $\sqrt{\frac{3}{2}}\mathrm{e}^{-\frac{1}{4}}$

SOLUTION: At $x = 0$, we have $y = \sqrt{1}\mathrm{e}^0 = 1$. Now for the slope:

$$f'(x) = \frac{1}{2}(x+1)^{-1/2}\mathrm{e}^{-x^2} + \sqrt{x+1}\mathrm{e}^{-x^2}(-2x)$$

so $f'(0) = \frac{1}{2}$, and $L(x) = 1 + \frac{1}{2}x$. Therefore, (note that the question asks you to approximate $f(1/2)$):

$$f(1/2) \approx L(1/2) = 1 + \frac{1}{2}\cdot\frac{1}{2} = \frac{5}{4}$$

10. Let $f(x) = \sqrt{x} - \frac{x}{3}$ on $[0, 9]$. Verify that the function satisfies all the hypotheses of the MVT, then find the values of c that satisfy its conclusion.

SOLUTION: f is continuous on the given interval, and $f'(x)$ does not exist at 0, but that's OK since for Rolle's Theorem, we need f to be differentiable on $(0, 9)$. Finally, check that $f(0) = f(9) = 0$. To find the value of c, Rolle's Theorem guarantees c so that $f'(c) = 0$:

$$f'(c) = \frac{1}{2\sqrt{c}} - \frac{1}{3} = 0 \quad \Rightarrow \quad c = \frac{9}{4} = 2.25$$

11. Let $f(x) = x^3 - 3x + 2$ on the interval $[-2, 2]$. Verify that the function satisfies all the hypotheses of the Mean Value Theorem, then find the values of c that satisfy its conclusion.

SOLUTION: f is a polynomial, so it is continuous and differentiable at all real numbers. Now we find c so that

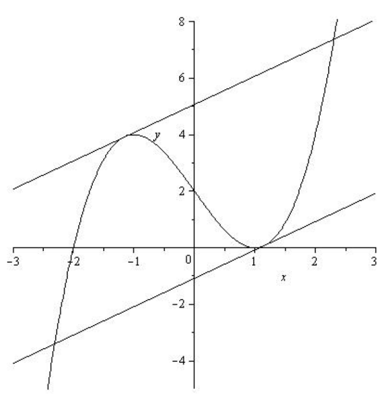
$$f'(c) = \frac{f(2)-f(-2)}{2-(-2)} = \frac{4-0}{4} = 1$$

And $f'(c) = 3c^2 - 3$, so:

$$3c^2 - 3 = 1 \quad \Rightarrow \quad c = \pm\frac{2}{\sqrt{3}}$$

Both of these are in the interval $[-2, 2]$.

Extra: The graph is below showing the tangent lines.



12. Let $f(x) = \tan(x)$. Show that $f(0) = f(\pi)$, but there is no number c in $(0, \pi)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

SOLUTION:

$$f(0) = \tan(0) = 0 \qquad f(\pi) = \tan(\pi) = \frac{\sin(\pi)}{\cos(\pi)} = \frac{0}{-1} = 0$$

Now, $f'(c) = \sec^2(x)$. But $\sec(x) = \frac{1}{\cos(x)}$ is always greater than (or equal to) 1. That is, there is no solution to:

$$\frac{1}{\cos^2(x)} = 0$$

This does not contradict Rolle's Theorem, since $\tan(x)$ is not continuous on $[0, \pi]$ (there is a vertical asymptote at $x = \pi/2$).

13. Find the limit, if it exists.

(a) This is a 0/0 form, so use l'Hospital's rule directly:

$$\lim_{x\to 0} \frac{\sin^{-1}(x)}{x} \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = 1$$

(b) This is a 0/0 form, so use l'Hospital's rule directly:

$$\lim_{x\to 0} \frac{x3^x}{3^x - 1} = \lim_{x\to 0} \frac{3^x + x3^x \ln(3)}{3^x \ln(3)} = \lim_{x\to 0} \frac{1 + x\ln(3)}{\ln(3)} = \frac{1}{\ln(3)}$$

(c) This is a product, $0 \cdot (-\infty)$, so convert to a fraction first:

$$\lim_{x\to 0^+} \sin(x)\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\csc(x)} = \lim_{x\to 0^+} \frac{1/x}{-\csc(x)\cot(x)}$$

To simplify the denominator, we see that $-\csc(x)\cot(x) = -\frac{\cos(x)}{\sin^2(x)}$, so that taking the reciprocal, and taking l'Hospital's rule one last time:

$$= \lim_{x\to 0^+} \frac{-\sin^2(x)}{x\cos(x)} = \lim_{x\to 0^+} \frac{-2\sin(x)\cos(x)}{\cos(x) - x\sin(x)} = \frac{0}{1} = 0$$

(d) Taking a cue from the way the problem is presented, we'll write this as a fraction taking the reciprocal of $\cot(2x)$:

$$\lim_{x\to 0} \cot(2x)\sin(6x) = \lim_{x\to 0} \frac{\sin(6x)}{\tan(2x)} = \lim_{x\to 0} \frac{6\cos(6x)}{2\sec^2(2x)} = \frac{6 \cdot 1}{2 \cdot 1^2} = 3$$

(e) We have the $f(x)^{g(x)}$ form- Take the limit of the log, and at the end, exponentiate.

$$\lim_{x\to 0^+} \sqrt{x}\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1/2}} = \lim_{x\to 0^+} \frac{1/x}{-(1/2)x^{-3/2}} = \lim_{x\to 0^+} -2\sqrt{x} = 0$$

Therefore,

$$\lim_{x\to 0^+} x^{\sqrt{x}} = e^0 = 1$$

(f) The idea is the same as before- take the limit of the log:

$$(4x+1)^{\cot(x)} \quad \to \quad \lim_{x\to 0^+} \cot(x)\ln(4x+1)$$

Before l'Hospital's rule, we need to rewrite the product as a fraction:

$$\lim_{x\to 0^+} \cot(x)\ln(4x+1) = \lim_{x\to 0^+} \frac{\ln(4x+1)}{\tan(x)} = \lim_{x\to 0^+} \frac{\frac{4}{4x+1}}{\sec^2(x)} = \frac{4}{1} = 4$$

So overall, $\lim_{x\to 0^+} (4x+1)^{\cot(x)} = e^4$.

14. Verify the given linear approximation (for small x).

   (a) $\sqrt[4]{1+2x} \approx 1 + \frac{1}{2}x$
   SOLUTION: We show that the tangent line to $f(x) = \sqrt[4]{1+2x}$ at $x = 0$ is $1+\frac{1}{2}x$. For the tangent line, the point on the curve is $(0, 1)$ and the slope is:

   $$f'(x) = \frac{1}{2}(1+2x)^{-3/4}\bigg|_{x=0} = \frac{1}{2}$$

   Therefore, the tangent line is:

   $$f(0) + f'(0)(x-0) = 1 + \frac{1}{2}x$$

   (b) $e^x \cos(x) \approx 1 + x$
   SOLUTION: We solve this the same way as the previous problem- Show that the tangent line to $f(x) = e^x \cos(x)$ at $x = 0$ is given by $1 + x$.
   For the line, the point is $(0, 1)$. The slope is:

   $$f'(x) = e^x \cos(x) - e^x \sin(x)\big|_{x=0} = 1 - 0 = 1$$

   Therefore, the tangent line is:

   $$f(0) + f'(0)(x-0) = 1 + x$$

15. At 1:00 PM, a truck driver picked up a fare card at the entrance of a tollway. At 2:15 PM, the trucker pulled up to a toll booth 100 miles down the road. After computing the trucker's fare, the toll booth operator summoned a highway patrol officer who issued a speeding ticket to the trucker. (The speed limit on the tollway is 65 MPH).

   (a) The trucker claimed that she hadn't been speeding. Is this possible? Explain.
   SOLUTION: Nope. Not possible. The trucker went 100 miles in 1.25 hours, which is not possible if you go (at a maximum) of 65 miles per hour (which would only get you (at a max) 81.25 miles). In terms of the MVT:

   $$\frac{\text{Change in Position}}{\text{Change in time}} = \frac{100}{1.25} = 80$$

   So we can guarantee that at some point in time, the trucker's speedometer read exactly 80 MPH.

   (b) The fine for speeding is \$35.00 plus \$2.00 for each mph by which the speed limit is exceeded. What is the trucker's minimum fine? By the last computation, the trucker had an *average* speed of 80 mph, so we can guarantee (by the MVT) that at some point, the speedometer read exactly 80. So, this gives \$35.00 + \$2.00 (15) = \$65.00

16. Let $f(x) = \frac{1}{x}$

   (a) What does the Extreme Value Theorem (EVT) say about f on the interval $[0.1, 1]$?
   SOLUTION: Since f is continuous on this closed interval, there is a global max and global min (on the interval).

   (b) Although f is continuous on $[1, \infty)$, it has no minimum value on this interval. Why doesn't this contradict the EVT?
   SOLUTION: The EVT was stated on an interval of the form $[a, b]$, which implies that we cannot allow a, b to be infinite.

17. Let f be a function so that $f(0) = 0$ and $\frac{1}{2} \leq f'(x) \leq 1$ for all x. Explain why $f(2)$ cannot be 3 (Hint: You might use a value theorem to help).

SOLUTION: We know that:

$$\frac{f(2) - f(0)}{2 - 0} = f'(c)$$

for some c in $(0, 2)$. Using the restrictions on the derivative,

$$\frac{1}{2} \leq \frac{f(2)}{2} \leq 1$$

so that $1 \leq f(2) \leq 2$.

18. Optimization extra practice:

(a) Suppose we have two numbers, one is a positive number and the other is its reciprocal. Find the two numbers so that the sum is small as possible.

SOLUTION: Let x and $1/x$ be the two numbers. Then we want to find the minimum of:

$$f(x) = x + \frac{1}{x} \qquad x \geq 0$$

If we look at the critical points of f, we see that $x = \pm 1$ (but we only take $x = 1$). Further, the derivative is:

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

We see that $f'(x) < 0$ if $0 < x < 1$ and $f'(x) > 0$ if $x > 1$. By the first derivative test, $x = 1$ is the minimum.

(b) Find two positive numbers such that their product is 16 and the sum is as small as possible.

SOLUTION: Let x, y be the two numbers. We want to find the minimum of $x + y$, if $xy = 16$. Turning this into a function of one variable using substitution: $y = 16/x$, we have:

$$f(x) = x + \frac{16}{x} \quad \Rightarrow \quad f'(x) = 1 - \frac{16}{x^2}$$

Therefore, $x = 4$ is the critical points, and by the first derivative test, we see that $f'(x) < 0$ if $0 < x < 4$ and $f'(x) > 0$ if $x > 4$, so the minimum value is $4 + \frac{16}{4} = 4 + 4 = 8$.

(c) A 20-inch piece of wire is bent into an L-shape. Where should the bend be made to minimize the distance between the two ends?

SOLUTION: Think of the wire as being bent so the corner is at the origin. Then the two ends of the wire are at $(x, 0)$ and $(0, y)$, and we can interpret the problem as saying that we want to find the minimum of $x^2 + y^2$ (the square of the distance) such that $x + y = 20$. Using substitution,

$$f(x) = x^2 + (20 - x)^2 \quad \Rightarrow \quad f'(x) = 2x - 2(20 - x) = 0 \quad \Rightarrow \quad 2x - 20 = 0 \quad \Rightarrow \quad x = 10$$

Now, we should also note that $0 \leq x \leq 20$, so that at the endpoints, $f(0) = 20^2$ and $f(20) = 20^2$. We have $f(10) = 10^2$ which gives us our (global) minimum. Therefore, L-shape should be a square, with each end being 10 inches.

(d) Find the point on the line $y = x$ closest to the point $(1, 0)$.

SOLUTION: We recall that it is a bit easier to minimize the *square* of the distance, so we'll find the minimum of

$$(x - 1)^2 + y^2 \text{ such that } y = x$$

Therefore, we want the minimum of

$$f(x) = (x - 1)^2 + x^2 = 2x^2 - 2x + 1 \quad \Rightarrow \quad f'(x) = 4x - 2 = 0 \quad \Rightarrow \quad x = 2$$

We also note that if $x < 2$, then $f'(x) < 0$ and if $x > 2$, then $f'(x) > 0$. Therefore, we have found the minimum at $x = 2$. The point on the line $y = x$ that is closest to $(1, 0)$ is the point $(2, 2)$.

(e) A box is constructed out of two different types of metal. The metal for the top and bottom, which are both square, costs \$1.00 per square foot, and the metal for the sides costs \$2.00 per square foot. Find the dimensions that minimize the cost of the box is the box must have a volume of 20 cubic feet.

SOLUTION: Since the top/bottom are square, let those dimensions be x, x and the height be h. The cost of the box is determined by the surface area- The top and bottom each have area x^2, and each side has area xh. Therefore,

$$\text{Cost } = 2x^2 \times \$1.00 + 4xh \times \$2.00 = 2x^2 + 8xh \text{ such that } V = x^2h = 20$$

Now we see that we can write the cost function as a function of x alone (using substitution)

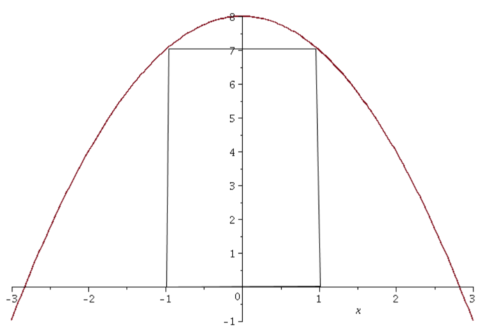
$$C(x) = 2x^2 + 8x\frac{20}{x^2} = 2x^2 + \frac{160}{x} \quad \Rightarrow \quad C'(x) = 4x - \frac{160}{x^2} = 0$$

Solving for x, we get

$$4x^3 = 160 \quad \Rightarrow \quad x^3 = 40 \quad \Rightarrow \quad x = \sqrt[3]{40}$$

And the height would be $20/x^2 = 20/(40^{2/3})$. We can check the first derivative as we did in the previous problems, C' changes sign from negative to positive at the critical point.

(f) A rectangle is to be inscribed between the $x-$axis and the upper part of the graph of $y = 8 - x^2$ (symmetric about the $y-$axis). For example, one such rectangle might have vertices: $(1,0), (1,7), (-1,7), (-1,0)$ which would have an area of 14. Find the dimensions of the rectangle that will give the largest area.



SOLUTION: Try drawing a picture first: The parabola opens down, goes through the $y-$intercept at 8, and has $x-$intercepts of $\pm\sqrt{8}$. Now, let x be as usual, so that the full length of the base of the rectangle is $2x$.

Then the height is y, or $8 - x^2$. Therefore, the area of the rectangle is:

$$A = 2xy = 2x(8 - x^2) = 16x - 2x^3$$

and $0 \le x \le \sqrt{8}$. We see that the area will be zero at the endpoints, so we expect a maximum at the critical point inside the interval:

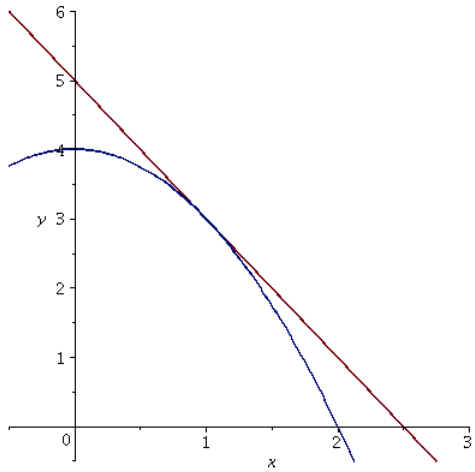
$$\frac{dA}{dx} = 16 - 6x^2$$

so the critical points are $x = \pm\sqrt{8/3}$, of which only the positive one is in our interval. So the dimensions of the rectangle are as follows (which give the maximum area of approx. 17.4):

$$2x = 2\sqrt{8}3 \qquad y = 8 - \frac{8}{3} = \frac{16}{3}$$

(g) What is the smallest possible area of the triangle that is cut off by the first quadrant and whose hypotenuse is tangent to the curve $y = 4 - x^2$ at some point?

SOLUTION: See the figure below.



The area of the triangle is $A = \frac{1}{2}bh$, where the base is the length of the $x-$ intercept of the tangent line and the height is the $y-$intercept of the tangent line.
If the $x-$coordinate for the tangent line is given by $(a, f(a))$ and the slope is $f'(a)$, then the equation of the line is:

$$y - f(a) = f'(a)(x - a)$$

Further, we have $f(a) = 4 - a^2$ and $f'(a) = -2a$. To find the $x-$intercept, set $y = 0$ and solve for x.

For the $x-$intercept, we get

$$0 - (4 - a^2) = -2a(x - a) \quad \Rightarrow \quad x = \frac{a^2 + 4}{2a}$$

For the $y-$intercept, we get

$$y - (4 - a^2) = -2a(0 - a) \quad \Rightarrow \quad y = a^2 + 4$$

Therefore, the area is given by

$$A = \frac{1}{2}\left(\frac{a^2+4}{2a}\right)(a^2 + 4) = \frac{(a^2+4)^2}{4a}$$

Set the derivative to zero:

$$\frac{dA}{da} = a^2 + 4 - \frac{(a^2+4)^2}{4a^2} = 0 \quad \Rightarrow \quad 1 = \frac{a^2+4}{4a^2} \quad \Rightarrow \quad 4a^2 = a^2 + 4$$

From which $a = 2/\sqrt{3}$. If we look at $a > 0$, this is the only critical point, with the derivative changing sign from negative to positive (so we have a global minimum). Therefore, the smallest possible area is the following (on the exam, you could leave this unevaluated to save time).

$$A = \frac{((4/3) + 4)^2}{4 \cdot 2/\sqrt{3}}$$

(h) You're standing with Elvis (the dog) on a straight shoreline, and you throw the stick in the water. Let us label as "A" the point on the shore closest to the stick, and suppose that distance is 7 meters. Suppose that the distance from you to the point A is 10 meters. Suppose that Elvis can run at 3 meters per second, and can swim at 2 meters per second. How far along the shore should Elvis run before going in to swim to the stick, if he wants to minimize the time it takes him to get to the stick?

SOLUTION: Recall that given distance d, rate r and time t, then $d = rt$, or $t = d/r$. If we let x be distance from point A that Elvis should travel before entering the water, then $10 - x$ is the distance that Elvis runs on the beach, and $\sqrt{x^2 + 49}$ is the distance Elvis has to swim. That gives the total time:

$$T(x) = \frac{\sqrt{x^2+49}}{2} + \frac{10 - x}{3} \qquad 0 \le x \le 10$$

We want to minimize T on the closed interval given, so first we find the critical points, then build a table. The critical points are:

$$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2+49}} - \frac{1}{3} = 0 \quad \Rightarrow \quad 3x = 2\sqrt{x^2+49} \quad \Rightarrow \quad x = \frac{14}{\sqrt{5}}$$

(we discard the negative solution). Now build a table:

x	0	$14/\sqrt{5}$	10
T	6.833	5.94	6.10

Therefore, Elvis minimizes his overall time by first running $10 - 14/\sqrt{5} \approx 3.74$ meters, then swimming the remaining 9.39 meters for a total of 6.10 seconds.

NOTE: Since you wouldn't have a calculator on the exam, I would try to make sure the numbers worked out better than this.

19. **Graphical Exercises**

The solutions to the odd problems are in the text.