Continuity and Differentiability Worksheet

(Be sure that you can also do the graphical exercises from the text- These were not included below! Typical problems are like problems 1-3, p. 161; 1-13, p. 171; 33-34, p. 172; 1-4, p. 131; 41, 46-48, 51 p. 176)

1. Finish the definition: A function f is said to be continuous at $x = a$ if:
2. The definition of continuity implies that we have three things to check. What are they?
3. Finish the definition: A function f is said to be right continuous at $x = a$ if:
4. Finish the definition: A function f is said to be continuous on the interval $[a, b]$ if:
5. Finish the definition: The derivative of f at $x = a$ is:
6. Finish the definition: A function f is said to be differentiable on the interval (a, b) if:
7. Why is the interval open in the last definition?
8. List three interpretations of the derivative of f at $x = a$.
9. True or False, and give a short reason:
   (a) If a function is differentiable, then it is continuous.
   (b) If a function is continuous, then it is differentiable.
   (c) If f is continuous on $[-1, 1]$ and $f(-1) = 4$ and $f(1) = 3$, then there is an $x = r$ so that $f(r) = \pi$.
   (d) If f is continuous at 5, and $f(5) = 2$, then the limit as $x \to 2$ of $f(4x^2 - 11)$ must be 2.
   (e) All functions are continuous on their domains.
   (f) It is possible for a function to be continuous everywhere, but not differentiable anywhere.
10. State the domain of each function, and say why the function is continuous on its domain:
    (a) $f(x) = \sqrt{\dfrac{4 - x^2}{1 - x^2}}$
    (b) $f(x) = \sin^{-1}(1 - x^2)$
    (c) $f(x) = \ln\left(\dfrac{x + 3}{x - 5}\right)$
    (d) $f(x) = \dfrac{x}{x^2 + 5x + 6}$
11. Explain why the function is discontinuous at the given point, $x = a$.
    (a) $f(x) = \ln|x + 3|$ at $a = -3$ (Extra: Is f continuous everywhere else?)
    (b)
    $$f(x) = \begin{cases} \frac{x^2-2x-8}{x-4}, & \text{if } x \neq 4 \\ 3, & \text{if } x = 4 \end{cases} \qquad a = 4$$
    (c) $f(x) = \frac{x^2-1}{x+1}$, at $a = -1$
    (d)
    $$f(x) = \begin{cases} 1 - x, & \text{if } x \leq 2 \\ x^2 - 2x, & \text{if } x > 2 \end{cases} \qquad a = 2$$
12. For each function, determine the value of the constant so that f is continuous everywhere:
    (a)
    $$f(x) = \begin{cases} \frac{x^2-16}{x-4}, & \text{if } x \neq 4 \\ C, & \text{if } x = 4 \end{cases}$$

(b)

$$f(x) = \begin{cases} 3x^2 - 1, & \text{if } x < 0 \\ cx + d, & \text{if } 0 \leq x \leq 1 \\ \sqrt{x+8}, & \text{if } x > 1 \end{cases}$$

(c)

$$f(x) = \begin{cases} \frac{\sqrt{7x+2} - \sqrt{6x+4}}{x-2}, & \text{if } x \geq -\frac{2}{7}, \text{ and } x \neq 2 \\ k, & \text{if } x = 2 \end{cases}$$

13. If f and g are continuous functions with $f(3) = 4$ and $\lim_{x \to 3} [2f(x) - g(x)] = 5$, what is $g(3)$?

14. Show that there must be at least one real solution to $x^5 - x^2 - 4 = 0$.

15. Each limit is the derivative of some function at some number a. State f and a in each case:

   (a) $\lim_{h \to 0} \dfrac{\sqrt{1+h} - 1}{h}$

   (b) $\lim_{x \to 1} \dfrac{x^9 - 1}{x - 1}$

   (c) $\lim_{t \to 0} \dfrac{\sin\left(\frac{\pi}{2} + t\right) - 1}{t}$

16. For each function below, compute the derivative using the definition. Also state the domain of the original function, and the domain of the derivative function.

   (a) $f(x) = \sqrt{1 + 2x}$

   (b) $g(x) = \frac{1}{x^2}$

   (c) $h(x) = x + \sqrt{x}$

   (d) $f(x) = \frac{2}{\sqrt{3-x}}$

   (e) $f(x) = \frac{x}{x^2-1}$

17. Let $f(x) = \sqrt[3]{x}$.

   (a) Use $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ to compute $f'(a)$, for $a \neq 0$. HINT: $x - a = (\sqrt[3]{x})^3 - (\sqrt[3]{a})^3$

   (b) Show that $f'(0)$ does not exist. What does this mean with respect to the graph of f at $a = 0$?

18. Given f below, where is f not continuous? Where is f not differentiable?

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 5 - x, & \text{if } 0 < x < 4 \\ \frac{1}{5-x}, & \text{if } x \geq 4 \end{cases}$$

19. Let $f(x) = x^3 - 2x$. (a) Find $f'(2)$. (b) Compute the equation of the line tangent to f at the point $(2, 4)$.

20. Sketch the graph of a function that satisfies the following conditions: $g(0) = 0$, $g'(0) = 3$, $g'(1) = 0$, $g'(2) = 1$

21. Find the slope of the line tangent to $y = x^2 + 2x$ at $x = -3$, then compute the equation of the line.