**Continuity and Differentiability Worksheet**

(Be sure that you can also do the graphical exercises from the text- These were not included below! Typical problems are like problems 1-3, p. 161; 1-13, p. 171; 33-34, p. 172; 1-4, p. 131; 41, 46-48, 51 p. 176)

1. Definition: A function $f$ is said to be continuous at $x = a$ if: $\lim_{x \to a} f(x) = f(a)$

2. The definition of continuity implies that we have three things to check. What are they? (1) $f(a)$ exists (or $f(a)$ is defined), (2) $\lim_{x \to a} f(x)$ exists. (3) The numbers in (1) and (2) are the same.

3. Finish the definition: A function $f$ is said to be right continuous at $x = a$ if: $\lim_{x \to a^+} f(x) = f(a)$.

4. Finish the definition: A function $f$ is said to be continuous on the interval $[a, b]$ if: $f$ is continuous for every point in $(a, b)$, is left continuous at $x = a$ and right continuous at $x = b$.

5. Finish the definition: The derivative of $f$ at $x = a$ is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{or} \quad \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

6. Finish the definition: A function $f$ is said to be differentiable on the interval $(a, b)$ if: $f$ is differentiable at each $x$ in the interval $(a, b)$.

7. Why is the interval open in the last definition? Because we need to be able to take the limit (from both sides) of each number in $(a, b)$. If we included the point $x = a$ or $x = b$, we'd have to take a "one-sided" derivative.

8. List three interpretations of the derivative of $f$ at $x = a$. (1) Slope of the Tangent Line to $f$ at $x = a$. (2) Velocity (if $f$ is a displacement function over time), (3) Instantaneous rate of change of $f$.

9. True or False, and give a short reason:

   (a) If a function is differentiable, then it is continuous. This is true (it was a theorem in class).

   (b) If a function is continuous, then it is differentiable. False. Example: $y = |x|$ at $x = 0$.

   (c) If $f$ is continuous on $[-1, 1]$ and $f(-1) = 4$ and $f(1) = 3$, then there is an $x = r$ so that $f(r) = \pi$. True- This is the statement of the Intermediate Value Theorem.

   (d) If $f$ is continuous at 5, and $f(5) = 2$, then the limit as $x \to 2$ of $f(4x^2 - 11)$ must be 2. True. Because $f$ is continuous, $\lim_{x \to a} f(x) = f(a)$.

   (e) All functions are continuous on their domains. Not True- All of our "basic" functions are, but there are functions that are not continuous anywhere (for example, the function that is zero on the rationals and 1 on the irrationals).

   (f) It is possible for a function to be continuous everywhere, but not differentiable anywhere. True- Many examples are called "fractals".

10. State the domain of each function, and say why the function is continuous on its domain:

    (a) $f(x) = \sqrt{\dfrac{4 - x^2}{1 - x^2}}$ This function will be continuous on its domain because it is constructed from the functions $4 - x^2$, $1 - x^2$ and $\sqrt{x}$. To find the domain, we require that:

$$\frac{(2+x)(2-x)}{(1+x)(1-x)} \geq 0$$

so we use a table to solve:

| | | | | | |
|---|---|---|---|---|---|
| $(2+x)$ | $-$ | $+$ | $+$ | $+$ | $+$ |
| $(2-x)$ | $+$ | $+$ | $+$ | $+$ | $-$ |
| $(1+x)$ | $-$ | $-$ | $+$ | $+$ | $+$ |
| $(1-x)$ | $+$ | $+$ | $+$ | $-$ | $-$ |
| | $x < -2$ | $-2 < x < -1$ | $-1 < x < 1$ | $1 < x < 2$ | $x > 2$ |

By the table, the domain is:
$x \leq -2$, or $-1 < x < 1$, or $x \geq 2$. which is also where $f(x)$ is continuous.

(b) $f(x) = \sin^{-1}(1 - x^2)$
First, recall that the domain of $\sin^{-1}(x)$ is $-1 \leq x \leq 1$. Therefore, the domain of $\sin^{-1}(1-x^2)$ is where $-1 \leq 1 - x^2 \leq 1$. This implies that $-2 \leq -x^2 \leq 0$, or where $0 \leq x^2 \leq 2$.
Thus, the answer is: The function $f$ is continuous on its domain, $-\sqrt{2} \leq x \leq \sqrt{2}$.

(c) $f(x) = \ln\left(\dfrac{x+3}{x-5}\right)$
This function will be continuous on its domain- The function $\ln(x)$ has a domain: $x > 0$, so $\ln(\frac{x+3}{x-5})$ has a domain that satisfies: $\frac{x+3}{x-5} > 0$. Use a table to solve:

| | | | |
|---|---|---|---|
| $x+3$ | $-$ | $+$ | $+$ |
| $x-5$ | $-$ | $-$ | $+$ |
| | $x < -3$ | $-3 < x < 5$ | $x > 5$ |

Conclusion: The function is continuous for $x < -3$ or $x > 5$.

(d) $f(x) = \dfrac{x}{x^2 + 5x + 6}$
Here, we need to make sure that $x^2 + 5x + 6 \neq 0$, so solve by factoring.
Conclusion: The function is continuous on all reals except where $x = -3$ and $x = -2$.

11. Explain why the function is discontinuous at the given point, $x = a$.

(a) $f(x) = \ln|x+3|$ at $a = -3$ (Extra: Is $f$ continuous everywhere else?)
$f$ is not continuous at $a = -3$ because $f$ is not defined for $a = -3$. (Yes, $f$ is continuous for all other $x$).

(b)
$$f(x) = \begin{cases} \frac{x^2-2x-8}{x-4}, & \text{if } x \neq 4 \\ 3, & \text{if } x = 4 \end{cases} \qquad a = 4$$

For this function, (1) $f$ is defined at $a = 4$, and $f(4) = 3$. (2) $\lim_{x\to 4} f(x) = \lim_{x \to 4} x + 2 = 6$. (3) The answers for (1) and (2) are not the same, so $f$ is not continuous at $a = 4$.

(c) $f(x) = \frac{x^2-1}{x+1}$, at $a = -1$
For this function, $f$ is not continuous at $a = -1$ because $f(-1)$ is not defined.

(d)
$$f(x) = \begin{cases} 1 - x, & \text{if } x \leq 2 \\ x^2 - 2x, & \text{if } x > 2 \end{cases} \qquad a = 2$$

We again check the three properties: (1) $f(2) = -1$, so $f$ is defined at $a = 2$. (2) $\lim_{x \to 2} f(x)$ does not exist- The limit coming from the right is 0, the limit coming from the left is $-1$. We don't have to check the third property- $f$ is not continuous because the limit at $a = 2$ does not exist.

12. For each function, determine the value of the constant so that $f$ is continuous everywhere:

(a)
$$f(x) = \begin{cases} \frac{x^2-16}{x-4}, & \text{if } x \neq 4 \\ C, & \text{if } x = 4 \end{cases}$$

First check the limit, since we want $C = \lim_{x \to 4} \dfrac{x^2 - 16}{x - 4} = 8$.

(b)

$$f(x) = \begin{cases} 3x^2 - 1, & \text{if } x < 0 \\ cx + d, & \text{if } 0 \leq x \leq 1 \\ \sqrt{x+8}, & \text{if } x > 1 \end{cases}$$

Again, we want to make sure the limits match at $x = 0$ and at $x = 1$ as we come in from the right and left.
At $x = 0$:

$$\lim_{x \to 0^+} f(x) = d \text{ and } \lim_{x \to 0^-} f(x) = -1$$

So $d = -1$. Next, we check $x = 1$:

$$\lim_{x \to 1^+} f(x) = 3 \text{ and } \lim_{x \to 1^-} f(x) = c + 1$$

so $c = 2$.

(c)

$$f(x) = \begin{cases} \frac{\sqrt{7x+2} - \sqrt{6x+4}}{x-2}, & \text{if } x \geq -\frac{2}{7}, \text{ and } x \neq 2 \\ k, & \text{if } x = 2 \end{cases}$$

We want the limits from the left and right of $x = 2$ to match up. First from the left:

$$\lim_{x \to 2^-} \frac{\sqrt{7x+2} - \sqrt{6x+4}}{x-2} = \lim_{x \to 2^-} \frac{\sqrt{7x+2} - \sqrt{6x+4}}{x-2} \cdot \frac{\sqrt{7x+2} + \sqrt{6x+4}}{\sqrt{7x+2} + \sqrt{6x+4}} =$$

$$\lim_{x \to 2^-} \frac{1}{\sqrt{7x+2} + \sqrt{6x+4}} = \frac{1}{8}$$

and, if we take the limit from the right:

$$\lim_{x \to 2^+} k = k$$

Putting the two together, $k = \frac{1}{8}$

13. If $f$ and $g$ are continuous functions with $f(3) = 4$ and $\lim_{x \to 3}[2f(x) - g(x)] = 5$, what is $g(3)$?

By continuity,

$$\lim_{x \to 3}[2f(x) - g(x)] = 2 \lim_{x \to 3} f(x) - \lim_{x \to 3} g(x) = 2f(3) - g(3) = 8 - g(3)$$

so that we now have:

$$8 - g(3) = 5$$

so $g(3) = 3$.

14. Show that there must be at least one real solution to $x^5 - x^2 - 4 = 0$. This is an Intermediate Value Theorem, where 0 is the Intermediate Value. Therefore, we need to find an $x$ so that $x^5 - x^2 - 4 < 0$ and an $x$ so that $x^5 - x^2 - 4 > 0$. For example, if $x = 1$, then we get a $-4$. If $x = 2$, we get $32 - 4 - 4 = 24 > 0$. Therefore, a solution to the equation is somewhere between $x = 1$ and $x = 2$.

15. Each limit is the derivative of some function at some number $a$. State $f$ and $a$ in each case:

(a) $\lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h}$
$f(x) = \sqrt{x}$ at $a = 1$.

(b) $\lim_{x \to 1} \frac{x^9 - 1}{x - 1}$
$f(x) = x^9$ at $a = 1$

(c) $\lim_{t\to 0} \dfrac{\sin\left(\frac{\pi}{2}+t\right)-1}{t}$
$f(x)=\sin(x)$ at $a=\frac{\pi}{2}$

16. For each function below, compute the derivative using the definition. Also state the domain of the original function, and the domain of the derivative function.

(a) $f(x)=\sqrt{1+2x}$ Domain of $f$: $x\geq -\frac{1}{2}$

$$\lim_{h\to 0}\frac{\sqrt{1+2(x+h)}-\sqrt{1+2x}}{h}=\lim_{h\to 0}\frac{\sqrt{1+2(x+h)}-\sqrt{1+2x}}{h}\cdot\frac{\sqrt{1+2(x+h)}+\sqrt{1+2x}}{\sqrt{1+2(x+h)}+\sqrt{1+2x}}=$$

$$\lim_{h\to 0}\frac{1+2x+2h-1-2x}{h(\sqrt{1+2(x+h)}+\sqrt{1+2x})}=\lim_{h\to 0}\frac{1}{(\sqrt{1+2(x+h)}+\sqrt{1+2x})}=\frac{1}{2\sqrt{1+2x}}$$

(b) $g(x)=\frac{1}{x^2}$

$$\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}=\lim_{h\to 0}\frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h}=\lim_{h\to 0}\frac{-2x-h}{x^2(x+h)^2}=-\frac{2}{x^3}$$

(c) $h(x)=x+\sqrt{x}$

$$\lim_{h\to 0}\frac{[x+h+\sqrt{x+h}]-[x+\sqrt{x}]}{h}=\lim_{h\to 0}\frac{h+\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\left(1+\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)=$$

$$1+\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}=1+\lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}=1+\frac{1}{2\sqrt{x}}$$

(d) $f(x)=\frac{2}{\sqrt{3-x}}$

$$\lim_{h\to 0}\frac{\frac{2}{\sqrt{3-(x+h)}}-\frac{2}{\sqrt{3-x}}}{h}=\lim_{h\to 0}\frac{2(\sqrt{3-x}-\sqrt{3-x-h})}{h\sqrt{3-x}\sqrt{3-x-h}}\cdot\frac{\sqrt{3-x}+\sqrt{3-x-h}}{\sqrt{3-x}+\sqrt{3-x-h}}=$$

$$\lim_{h\to 0}\frac{2}{\sqrt{3-x}\sqrt{3-x-h}(\sqrt{3-x}+\sqrt{3-x-h})}=\frac{2}{(3-x)2\sqrt{3-x)}}=\frac{1}{(3-x)^{3/2}}$$

(e) $f(x)=\frac{x}{x^2-1}$

$$\lim_{h\to 0}\frac{\frac{x+h}{(x+h)^2-1}-\frac{x}{x^2-1}}{h}=\lim_{h\to 0}\frac{(x+h)(x^2-1)-x((x+h)^2-1)}{h((x+h)^2-1)(x^2-1)}=\lim_{h\to 0}\frac{-hx^2+xh^2-h}{h((x+h)^2-1)(x^2-1)}=$$

$$\frac{-x^2-1}{(x^2-1)^2}$$

17. Let $f(x)=\sqrt[3]{x}$.

(a) Use $\lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}$ to compute $f'(a)$, for $a\neq 0$.

$$\lim_{x\to a}\frac{\sqrt[3]{x}-\sqrt[3]{a}}{x-a}=\lim_{x\to a}\frac{\sqrt[3]{x}-\sqrt[3]{a}}{\left(\sqrt[3]{x}-\sqrt[3]{a}\right)\left(x^{2/3}+a^{1/3}x^{1/3}+a^{2/3}\right)}=\lim_{h\to 0}\frac{1}{x^{2/3}+a^{2/3}x^{2/3}+a^{2/3}}=$$

$$\frac{1}{3x^{2/3}}$$

(b) Show that $f'(0)$ does not exist. What does this mean with respect to the graph of $f$ at $a=0$?
From our formula, we see that, as $x\to 0$, $f'(x)\to\infty$, which means that at $x=0$, there is a vertical tangent line.

18. Given $f$ below, where is $f$ not continuous? Where is $f$ not differentiable?

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 5 - x, & \text{if } 0 < x < 4 \\ \frac{1}{5-x}, & \text{if } x \geq 4 \end{cases}$$

Not continuous at: $x = 0$, because the limit from the right is not the limit from the left. It is continuous at $x = 4$, and for all other $x$.

The function is not differentiable at $x = 0$, because it was not continuous there. Let's see if its differentiable at $x = 4$. We need to see if the limit from the left and right of $\dfrac{f(x+h) - f(x)}{h}$ are the same.

From the left, the limit is $-1$, and from the right, the limit is 1, so the function is not differentiable at $x = 4$.

Details:

$$\lim_{h \to 0^-} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0^-} \frac{1 - h - 1}{h} = -1$$

$$\lim_{h \to 0^+} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0^+} \frac{\frac{1}{1-h} - 1}{h} = \lim_{h \to 0^+} \frac{1 - 1 + h}{h(1-h)} = 1$$

19. Let $f(x) = x^3 - 2x$. (a) Find $f'(2)$. (b) Compute the equation of the line tangent to $f$ at the point $(2, 4)$.

$$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(2+h)^3 - 2(2+h) - (2^3 - 2(2))}{h} = \lim_{h \to 0} \frac{h(10 + 6h + h^2)}{h} = 10$$

20. Sketch the graph of a function that satisfies the following conditions: $g(0) = 0$, $g'(0) = 3$, $g'(1) = 0$, $g'(2) = 1$

Your graph should have at least: A point at $(0, 0)$ with the curve going through the origin fairly steeply (local slope of 3), where the curve goes through $x = 1$, the curve should be flat (slope of zero), and finally, where the curve goes through $x = 2$, the slope should be about 1.

21. Find the slope of the line tangent to $y = x^2 + 2x$ at $x = -3$, then compute the equation of the line.

$$m_{\tan} = \lim_{h \to 0} \frac{f(-3+h) - f(-3)}{h} = \lim_{h \to 0} \frac{(-3+h)^2 + 2(-3+h) - 3}{h} = \lim_{h \to 0} \frac{-4h + h^2}{h} = -4$$

The tangent line has the equation: (In general: $y - f(a) = f'(a)(x - a)$)

$$y - 3 = -4(x + 3)$$