Sample Exam Questions

Disclaimer: These questions are intended to give you a variety of typical-to-harder type questions, and to look at questions "context-free".

1. Find the local maximums and minimums: $f(x) = x^3 - 3x + 1$ Show your answer is correct by using both the first derivative test and the second derivative test.

   $f'(x) = 3x^2 - 3$, so $f'(x) = 0$ if $x = \pm 1$. To use the first derivative test, build a table to determine where $f'(x) = 3(x^2 - 1)$ is positive/negative. You should find that $f'(x) > 0$ if $x < -1$ and $x > 1$, and $f'(x) < 0$ if $-1 < x < 1$. Therefore, there is a local maximum at $x = -1$, and a local minimum at $x = 1$. There is no global max or min.

   The second derivative test: $f''(x) = 6x$, so at $x = 1$, the function is concave up, so there is a local minimum. At $x = -1$, the function is concave down, so there is a local maximum.

2. Find dy in terms of x and dx: $y = \frac{x}{\sin(2x)}$.

$$dy = \frac{1 \cdot \sin(2x) - 2x\cos(2x)}{\sin^2(2x)}\, dx$$

3. Compute the derivative of y with respect to x:

   (a) $y = \sqrt[3]{2x+1}\sqrt[5]{3x-2}$

   Using logarithmic differentiation:

$$\ln(y) = \frac{1}{3}\ln(2x+1) + \frac{1}{5}\ln(3x-2)$$

$$\frac{1}{y}y' = \frac{1}{3}\cdot\frac{2}{2x+1} + \frac{1}{5}\cdot\frac{3}{3x-2}$$

$$y' = \sqrt[3]{2x+1}\sqrt[5]{3x-2}\left(\frac{1}{3}\cdot\frac{2}{2x+1} + \frac{1}{5}\cdot\frac{3}{3x-2}\right)$$

   Using just the product rule:

$$y' = \frac{2}{3}(2x+1)^{-2/3}\sqrt[5]{3x-2} + \sqrt[3]{2x+1}\cdot\frac{3}{5}\cdot(3x-2)^{-4/5}$$

   (b) $y = \frac{1}{1+u^2}$, where $u = \frac{1}{1+x^2}$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{-2u}{(1+u^2)^2}\frac{-2x}{(1+x^2)^2}$$

   Back-substituting $u = \frac{1}{1+x^2}$,

$$\frac{-2\cdot\frac{-2x}{(1+x^2)^2}}{\left(1+\left(\frac{-2x}{(1+x^2)^2}\right)^2\right)^2}\frac{-2x}{(1+x^2)^2} = \frac{-8x^2}{(1+x^2)^4 + 4x^2}$$

(c) $\sqrt[3]{y} + \sqrt[3]{x} = 4xy$

$$\frac{1}{3}y^{-2/3}\ y' + \frac{1}{3}x^{-2/3} = 4y + 4xy'$$

$$\left(\frac{1}{3y^{2/3}} - 4x\right) y' = 4y - \frac{1}{3x^{2/3}}$$

$$y' = \frac{4y - \frac{1}{3x^{2/3}}}{\frac{1}{3y^{2/3}} - 4x} = \frac{12yx^{2/3} - 1}{1 - 12xy^{2/3}} \cdot \frac{x^{2/3}}{y^{2/3}}$$

(d) $\sqrt{x+y} = \sqrt[3]{x-y}$

$$\frac{1}{2}(x+y)^{-1/2}(1+y') = \frac{1}{3}(x-y)^{-2/3}(1-y')$$

$$\frac{1}{2}(x+y)^{-1/2} + \frac{1}{2}(x+y)^{-1/2}y' = \frac{1}{3}(x-y)^{-2/3} - \frac{1}{3}(x-y)^{-2/3}y'$$

$$(\frac{1}{2}(x+y)^{-1/2} + \frac{1}{3}(x-y)^{-2/3})y' = \frac{1}{2}(x+y)^{-1/2} - \frac{1}{3}(x-y)^{-2/3}$$

$$y' = \frac{\frac{1}{2}(x+y)^{-1/2} - \frac{1}{3}(x-y)^{-2/3}}{\frac{1}{2}(x+y)^{-1/2} + \frac{1}{3}(x-y)^{-2/3}}$$

(e) $y = \sin(2\cos(3x))$

$$y' = \cos(2\cos(3x)) \cdot -2\sin(3x) \cdot 3 = -6\cos(2\cos(3x))\sin(3x)$$

(f) $y = (\cos(x))^{2x}$

$$\ln(y) = 2x\ln(\cos(x)) \Rightarrow \frac{y'}{y} = 2\ln(\cos(x)) + \frac{-2x\sin(x)}{\cos(x)}$$

$$y' = (\cos(x))^{2x}(2\ln(\cos(x)) - 2x\tan(x))$$

(g) $y = (\tan^{-1}(x))^{-1}$

$$y' = -1 \cdot (\tan^{-1}(x))^{-2} \cdot \frac{1}{1+x^2} = \frac{-1}{(\tan^{-1}(x))^2 \cdot (1+x^2)}$$

(h) $y = \sin^{-1}(\cos^{-1}(x))$

$$y' = \frac{1}{\sqrt{1-(\cos^{-1}(x))^2}} \cdot \frac{-1}{\sqrt{1-x^2}}$$

(i) $y = \log_{10}(x^2 - x)$ Recall that $\log_a(b) = \frac{\ln(b)}{\ln(a)}$, so $y = \frac{\ln(x^2-x)}{\ln(10)}$, and

$$y' = \frac{1}{\ln(10)} \cdot \frac{1}{x^2-x} \cdot (2x-1) = \frac{2x-1}{\ln(10)(x^2-x)}$$

(j) $y = x^{x^2+2}$ Use logarithmic differentiation:

$$\ln(y) = (x^2+2)\ln(x) \Rightarrow \frac{y'}{y} = 2x\ln(x) + (x^2+2)\frac{1}{x}$$

$$y' = x^{x^2+2}\left(2x\ln(x) + \frac{x^2+2}{x}\right)$$

4. List the three items we need to check to see if a function $f(x)$ is continuous at $x = a$. (a) $f(a)$ is defined. (b) $\lim_{x\to a} f(x)$ exists. (c) Parts (a) and (b) are equal.

5. Derive the formula for the derivative of $y = \sec^{-1}(x)$. First, set $\sec(y) = x$. Draw the appropriate triangle. If y is an angle, then the hypotenuse is x and the side adjacent is 1. This leaves the side opposite as $\sqrt{x^2-1}$. Now,

$$\sec(y)\tan(y)y' = 1 \Rightarrow y' = \frac{1}{\sec(y)\tan(y)} = \frac{1}{x\sqrt{x^2-1}}$$

   Where the last step is obtained by using the triangle.

6. Write the equation of the line tangent to $x = \sin(2y)$ at $x = 1$. We need a point and a slope. The point is $x = 1$, so $1 = \sin(2y)$.

   Solve $1 = \sin(2y)$ by looking at the sine function. It is 1 when $2y = \frac{\pi}{2}$ (The problem should have stated that this would be the only y we need to consider).

   Now, $y = \frac{\pi}{4}$.

   The slope is determined by differentiating:

$$1 = \cos(2y)2y' \Rightarrow y' = \frac{1}{2\cos(2y)}$$

   which does not exist at $y = \frac{\pi}{4}$. You could either answer that the derivative does not exist, or that there is a vertical tangent line at $x = 1$ (which *is* the equation).

7. If a hemispherical bowl with radius 1 foot is filled with water to a depth of x inches, then the volume of the water in the bowl is given by:

$$V = \frac{\pi}{3}\left(36x^2 - x^3\right) \quad \text{cubic inches}$$

   If the water flows out a hole in the bottom at the rate of 36π cubic inches per second, how fast is the water level decreasing when $x = 6$ inches?

The question is asking us to compute $\frac{dx}{dt}$ when $x = 6$ and $\frac{dV}{dt} = -36\pi$ (Negative because the volume is decreasing).

$$\frac{dV}{dt} = \frac{\pi}{3}(72x \cdot \frac{dx}{dt} - 3x^2 \cdot \frac{dx}{dt})$$

$$-36\pi = \frac{\pi}{3}(432 - 108)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{-1}{3}$$

8. Compute the limit, if it exists. You may use any method (except a numerical table).

(a) $\lim_{x\to 0} \frac{x - \sin(x)}{x^3}$ Use L'Hospital's Rule:

$$\lim_{x\to 0} \frac{x - \sin(x)}{x^3} = \lim_{x\to 0} \frac{1 - \cos(x)}{3x^2} = \lim_{x\to 0} \frac{\sin(x)}{6x} = \lim_{x\to 0} \frac{\cos(x)}{6} = \frac{1}{6}$$

(b) $\lim_{x\to 4^+} \frac{x-4}{|x-4|}$ If $x > 4$, then $|x - 4| = x - 4$, so the limit is 1.

(c) $\lim_{x\to -\infty} \sqrt{\frac{2x^2 - 1}{x + 8x^2}}$

$$\lim_{x\to -\infty} \sqrt{\frac{2x^2 - 1}{x + 8x^2}} = \lim_{x\to -\infty} \sqrt{\frac{2 - \frac{1}{x^2}}{\frac{1}{x} + 8}} = \frac{1}{4}$$

Note: We're dividing the numerator AND the denominator by $-\sqrt{x^2}$, so the negative signs cancelled.

(d) $\lim_{x\to\infty} \sqrt{x^2 + x + 1} - \sqrt{x^2 - x}$

$$\lim_{x\to\infty} \sqrt{x^2 + x + 1} - \sqrt{x^2 - x} \cdot \frac{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}}$$

$$\lim_{x\to\infty} \frac{x^2 + x + 1 - x^2 + x}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} = \lim_{x\to\infty} \frac{2x + 1}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}}$$

$$\lim_{x\to\infty} \frac{2 + \frac{1}{x}}{\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x}}} = 1$$

(e) $\lim_{h\to 0} \frac{(1+h)^{-2} - 1}{h} = \lim_{h\to 0} \frac{\frac{1}{(1+h)^2} - \frac{(1+h)^2}{(1+h)^2}}{h} =$

$$\lim_{h\to 0} \frac{1 - (1 + 2h + h^2)}{h(1+h)^2} = -2$$

(f) $\lim_{x\to 1} \frac{x^{1000}-1}{x-1} = \lim_{x\to 1} \frac{1000x^{99}}{1} = 1000$

(g) $\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)} = \lim_{x\to 0} \frac{1}{\left(\frac{4}{1+16x^2}\right)} = \frac{1}{4}$

(h) $\lim_{x\to 1} x^{\frac{1}{1-x}} = \mathrm{e}^{\lim_{x\to 1} \ln(x^{\frac{1}{1-x}})} = \mathrm{e}^{\lim_{x\to 1} \frac{1}{1-x}\ln(x)}$ Now we have:

$$\lim_{x\to 1} \frac{\ln(x)}{x-1} = \lim_{x\to 1} \frac{\frac{1}{x}}{-1} = -1$$

so overall,

$$\lim_{x\to 1} x^{\frac{1}{1-x}} = \mathrm{e}^{-1}$$

9. Determine all vertical/horizontal asymptotes and critical points of $f(x) = \frac{2x^2}{x^2-x-2}$

Vertical asymptotes: $x-$values where the denominator is zero, and the numerator is not.

$$x^2 - x - 2 = 0 \Rightarrow x = 2, -1$$

The vertical asymptotes are: $x = 2$ and $x = -1$.

Horizontal Asymptotes:

$$\lim_{x\to\infty} \frac{2x^2}{x^2-x-2} = \lim_{x\to\infty} \frac{2}{1-\frac{1}{x}-\frac{2}{x^2}} = 2$$

So $y = 2$ is the horizontal asymptote. Note that the limit as x goes to minus infinity will give the same value.

10. (**Won't be on the exam, but interesting to look at!**) Why does Newton's Method fail, if:

(a) $f(x) = x^2 + 1$

(b)

$$f(x) = \begin{cases} \sqrt{x}, & \text{if } x \geq 0 \\ -\sqrt{-x}, & \text{if } x < 0 \end{cases}$$

11. Find values of m and b so that (1) f is continuous, and (2) f is differentiable.

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ mx + b & \text{if } x > 2 \end{cases}$$

For f to be continuous at $x = 2$, $\lim_{x\to 2} f(x) = f(2)$, so that

$$2^2 = 2m + b \Rightarrow 4 = 2m + b$$

For f to be differentiable at $x = 2$, the slopes need to match at $x = 2$:

$$2(2) = m$$

Putting these together, $m = 4$ and $b = -4$.

12. Find the local and global extreme values of $f(x) = \frac{x}{x^2+x+1}$ on the interval $[-2, 0]$.

    First, make sure f is continuous on $[-2, 0]$. It is, since the denominator is never zero (use the quadratic formula on $x^2 + x + 1 = 0$ to check).

    Next, check the endpoints: If $x = -2, y = \frac{-2}{3}$, if $x = 0, y = 0$.

    Check the critical points:

$$f'(x) = \frac{x^2 + x + 1 - x(2x + 1)}{(x^2 + x + 1)^2} = 0$$

    $x = \pm 1$. Since $x = 1$ is outside the interval, we consider only $x = -1$. If $x = -1$, $y = -1$.

    We now have that f reaches its minimum at $x = -1$ of $y = -1$.

    f reaches its maximum at $x = 0$ of $y = 0$.

13. Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{9} + \frac{y^2}{25} = 1$

    Note that x must be in the interval $-3 \leq x \leq 3$. We'll take the positive side for half the width of the rectangle, so that x must be in the interval $[0, 3]$. At both endpoints, the rectangle has zero area.

    To find half the height, solve the elliptic equation for y:

$$y^2 = \frac{25}{9}(9 - x^2) \rightarrow y = \frac{5}{3}\sqrt{9 - x^2}$$

    The area of the rectangle is then:

$$A(x) = 2x \cdot 2\frac{5}{3}\sqrt{9 - x^2} = \frac{20}{3}x\sqrt{9 - x^2}$$

    To find the max, set $A'(x) = 0$, solve for x:

$$A'(x) = \frac{20}{3}\left(\sqrt{9 - x^2} + x\frac{1}{2}(9 - x^2)^{-1/2}(-2x)\right) = \frac{20}{3}\left(\sqrt{9 - x^2} - \frac{x^2}{\sqrt{9 - x^2}}\right)$$

    Setting $A'(x) = 0$, we get:

$$\sqrt{9 - x^2} = \frac{x^2}{\sqrt{9 - x^2}} \Rightarrow 9 - x^2 = x^2 \Rightarrow 10x^2 = 9 \Rightarrow x = \frac{3}{\sqrt{10}}$$

    To find the maximum area, put this x into $A(x)$ to get:

$$A(3/\sqrt{10}) = 18$$

14. Suppose f is differentiable so that:

$$f(1) = 1,\ f(2) = 2,\ f'(1) = 1\ f'(2) = 2$$

If $g(x) = f(x^3 + f(x^2))$, evaluate $g'(0)$.

$$g'(x) = f'(x^3 + f(x^2)) \cdot \left(3x^2 + f'(x^2) \cdot 2x\right)$$

so

$$g'(0) = f(0 + f(0))(0 + f'(0) \cdot 0) = 0$$

This was in error... I meant for you to compute $g'(1)$, which is:

$$g'(1) = f'(1 + f(1))(3 + f'(1) \cdot 2) = f'(2)(3 + 2) = 10$$

15. Find y'' by implicit differentiation:

$$x^2 + 6xy + y^2 = 8$$

$$2x + 6y + 6xy' + 2yy' = 0 \Rightarrow y' = \frac{-2x - 6y}{6x + 2y}$$

$$2 + 6y' + 6y' + 6xy'' + 2y'y' + 2yy'' = 0$$

$$y'' = \frac{-2 - 12y' - 2(y')^2}{6x + 2y}$$

Extra Credit: Try simplifying by substituting the expression we got for y'!

16. Let

$$x^2y + a^2xy + \lambda y^2 = 0$$

(a) Let a and λ be constants, and let y be a function of x. Calculate $\frac{dy}{dx}$:

$$2xy + x^2y' + a^2y + a^2xy' + 2\lambda yy' = 0$$

$$y'(x^2 + a^2x + 2\lambda y) = -2xy - a^2y \Rightarrow y' = \frac{-2xy - a^2y}{x^2 + a^2x + 2\lambda y}$$

(b) Let x and y be constants, and let a be a function of λ. Calculate $\frac{da}{d\lambda}$:

$$2aa'xy + y^2 = 0 \Rightarrow \frac{da}{d\lambda} = \frac{-y^2}{2axy} = \frac{-y}{2ax}$$

17. Show that $x^4 + 4x + c = 0$ has at most one solution in the interval $[-1, 1]$.

CORRECTION: The equation should be $x^5 - 6x + c = 0$

First, if $x = -1$, then $f(-1) = c - 7$. If $x = 1$, $f(1) = c - 5$. This says that if $5 < c < 7$, then there is a solution, otherwise there might be no solution.

Suppose that $f(x) = 0$ somewhere between $-1 < x < 1$. Then there must be an x in the interval $(-1, 1)$ so that:

$$f'(x) = \frac{f(-1) - f(1)}{-1 - 1} = \frac{c - 7 - (c - 5)}{-2} = 1$$

However, $f'(x) = 5x^4 - 6$, then $5x^4 - 6 = 1$, so $x = 1$, which is outside our interval for x.

Therefore, if there is some solution to $f(x) = 0$ in $[-1, 1]$, there is at most one solution in $[-1, 1]$.

18. True or False, and give a short explanation.

    (a) If $f'(r)$ exists, then
    $$\lim_{x \to r} f(x) = f(r)$$
    True. Note that the limit above is the definition of what it means for f to be continuous at r. If $f'(r)$ exists, then f is differentiable at r. If f is differentiable at r, it must be continuous at r.

    (b) If f and g are differentiable, then:
    $$\frac{d}{\mathrm{dx}}(f(g(x)) = f'(x)g'(x)$$
    False. This is not the chain rule: $f'(g(x))g'(x)$.

    (c) If $f(x) = x^2$, then the equation of the tangent line at $x = 3$ is:
    $$y - 9 = 2x(x - 3)$$
    False. The equation above is the equation of a parabola, not a line. To get the equation of the tangent line, you must put $x = 3$ into the derivative to get a *number* for the slope.

    (d)
    $$\lim_{\theta \to \frac{\pi}{3}} \frac{\cos(\theta) - \frac{1}{2}}{\theta - \frac{\pi}{3}} = -\sin\left(\frac{\pi}{3}\right)$$
    True. You can argue a couple of different ways. For example, use L'Hospital's rule, or see that the expression is the derivative of $\cos(\theta)$ at $\theta = \frac{\pi}{3}$.

    (e) There is no solution to $\mathrm{e}^x = 0$

    True. The range of e^x is $(0, \infty)$. You can also argue this graphically.

(f) $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$
False. For sine to be invertible, we have to restrict its domain to $\theta \in [-\pi/2, \pi/2]$. Use the unit circle to see that this will be $\frac{-\pi}{3}$.

(g) $5^{\log_5(2x)} = 2x$, for $x > 0$.
True. $A^{\log_A(x)} = x$, since A^x and $log_A(x)$ are inverse functions.

(h) $\frac{d}{dx}\ln(|x|) = \frac{1}{x}$, for all $x \neq 0$.
True. Recall that $\frac{d}{dx}|x| = \frac{x}{|x|}$, so that

$$\frac{d}{dx}\ln(|x|) = \frac{1}{|x|} \cdot \frac{x}{|x|} = \frac{1}{x}$$

(i) $\frac{d}{dx}10^x = x10^{x-1}$
False. $\frac{d}{dx}10^x = 10^x \cdot \ln(10)$

(j) If $x > 0$, then $(\ln(x))^6 = 6\ln(x)$
False. $\ln(x^6) = 6\ln(x)$

19. Find the domain of $\ln(x - x^2)$:

Use a table to find where $x - x^2 = x(1-x) > 0$. Using the table, we find that $x < 0$ or $x > 1$.

20. Find the value of c guaranteed by the Mean Value Theorem, if $f(x) = \frac{x}{x+2}$ on the interval $[1, 4]$.

We wish to find c so that:

$$f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{\frac{2}{3} - \frac{1}{3}}{3} = \frac{1}{9}$$

Now compute $f'(c)$ and solve for c:

$$\frac{2}{(c+2)^2} = \frac{1}{9}$$

so $c = -2 \pm 3\sqrt{2}$. Choose the one in the right interval, $c = -2 + 3\sqrt{2}$.

21. Compute the limit, without using L'Hospital's Rule. $\lim_{x\to 7} \frac{\sqrt{x+2} - 3}{x - 7}$

$$\lim_{x\to 7} \frac{\sqrt{x+2} - 3}{x - 7} \cdot \frac{\sqrt{x+2} + 3}{\sqrt{x+2} + 3} = \lim_{x\to 7} \frac{x + 2 - 9}{(x-7)(\sqrt{x+2} + 3)} = \frac{1}{6}$$

22. For what value(s) of c does $f(x) = cx^4 - 2x^2 + 1$ have both a local maximum and a local minimum?

We want $f'(x) = 0$ to have two solutions. Taking the derivative, we see that:

$$4cx^3 - 4x = 0 \Rightarrow 4x(cx^2 - 1) = 0$$

so we need $cx^2 - 1 = 0$ to have a solution, which it will if $c > 0$.

23. If $f(x) = \sqrt{1-2x}$, determine $f'(x)$ *by using the definition of the derivative.*

Recall that:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

so, computing these quantities, we get:

$$\lim_{h \to 0} \frac{\sqrt{1-2x-2h} - \sqrt{1-2x}}{h} =$$

$$\lim_{h \to 0} \frac{\sqrt{1-2x-2h} - \sqrt{1-2x}}{h} \cdot \frac{\sqrt{1-2x-2h} + \sqrt{1-2x}}{\sqrt{1-2x-2h} + \sqrt{1-2x}} =$$

$$\lim_{h \to 0} \frac{-2h}{h(\sqrt{1-2x-2h} + \sqrt{1-2x})} = \frac{-2}{2\sqrt{1-2x}} = \frac{-1}{\sqrt{1-2x}}$$

24. Use Newton's Method to find the absolute minimum of $f(x) = x^6 + 2x^2 - 8x + 3$ correct to four decimal places.

This won't be tested on the Final.

25. A *point of inflection* for a function f is the x value for which $f''(x)$ changes sign (either from positive to negative or vice versa).

   (a) If f'' is continuous, then $f''(x) = 0$ at an inflection point. What theorem did we have that proves this?

   The Intermediate Value Theorem, applied to $f''(x)$.

   (b) Find constants a and b so that $(1, 6)$ is an inflection point for $y = x^3 + ax^2 + bx + 1$.

$$y' = 3x^2 + 2ax + b$$

$$y'' = 6x + 2a$$

   At $x = 1$, we want $y'' = 0$, so $a = -3$. We also want $f(1) = 6$, so

$$6 = 1 + (-3) + b + 1$$

   and $b = 7$.

26. Suppose that $F(x) = f(g(x))$ and $g(3) = 6$, $g'(3) = 4$, $f(3) = 2$ and $f'(6) = 7$. Find $F'(3)$.

First, $F'(x) = f'(g(x))g'(x)$, so $F'(3) = f'(g(3))g'(3) = f'(6) \cdot 4 = 7 \cdot 4 = 28$.

27. Let $G(x) = h(\sqrt{x})$. Then where is G differentiable? Find $G'(x)$.

Addition to problem: Assume $h(x)$ is differentiable everywhere.

$G'(x) = h'(\sqrt{x})\frac{1}{2}x^{-1/2} = \frac{h'(\sqrt{x})}{2\sqrt{x}}$, so G is differentiable for $x > 0$.

28. If position is given by: $f(t) = t^4 - 2t^3 + 2$, find the times when the acceleration is zero. Then compute the velocity at these times.

$f'(t) = 4t^3 - 6t^2$, and $f''(t) = 12t^2 - 12t$. The acceleration is zero at $t = 0$ and $t = 1$. $f'(0) = 0$ and $f'(1) = -2$.

29. If $y = \sqrt{5t - 1}$, compute y'''.

$$y' = \frac{5}{2}(5t-1)^{-1/2}, \quad y'' = \frac{-25}{4}(5t-1)^{-3/2} \quad , y''' = \frac{375}{8}(5t-1)^{-5/2}$$

30. Find a second degree polynomial so that $P(2) = 5$, $P'(2) = 3$, and $P''(2) = 2$.

The general second degree polynomial is $P(x) = ax^2 + bx + c$. $P'(x) = 2ax + b$, and $P''(x) = 2a$. We want $P''(2) = 2$, so $a = 1$.

Now, $P'(2) = 3$, so $2 + b = 3$, and $b = 1$. Finally, $P(2) = 5$, so $4 + 2 + c = 5$, so $c = -1$.

In conclusion, $P(x) = x^2 + x - 1$.

31. If $f(x) = (2 - 3x)^{-1/2}$, find $f(0)$, $f'(0)$, $f''(0)$.

$$f(x) = (2-3x)^{-1/2}, \quad f'(x) = \frac{3}{2}(2-3x)^{-3/2}, \quad f''(x) = \frac{27}{4}(2-3x)^{-5/2}$$

$$f(0) = \frac{1}{\sqrt{2}}, \quad f'(0) = \frac{3}{4\sqrt{2}}, \quad f''(0) = \frac{27}{16\sqrt{2}}$$

32. Car A is traveling west at 50 mi/h, and car B is traveling north at 60 mi/h. Both are headed for the intersection between the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

Let $A(t)$, $B(t)$ be the positions of cars A and B at time t. Let the distance between them be $z(t)$, so that the Pythagorean Theorem gives:

$$z^2 = A^2 + B^2$$

Translating the question, we get that we want to find $\frac{dz}{dt}$ when $A = 0.3$, $B = 0.4$, (so $z = 0.5$), $A'(t) = 50$, $B'(t) = 60$. Then:

$$2z\frac{dz}{dt} = 2A\frac{dA}{dt} + 2B\frac{dB}{dt}$$

Putting in the numbers,

$$2 \cdot 0.5 \cdot \frac{dz}{dt} = 2 \cdot 0.3 \cdot 50 + 2 \cdot 60$$

and solve for $\frac{dz}{dt}$, 78.

33. Compute Δy and dy for the value of x and Δx: $f(x) = 6 - x^2$, $x = -2$, $\Delta x = 0.4$.

$$\Delta y = f(-2 + 0.4) - f(-2) = 3.44 - 2 = 1.44$$

$$dy = f'(x)\,dx = (-4)(0.4) = 1.6$$

34. Find the linearization of $f(x) = \sqrt{1 - x}$ at $x = 0$.

To linearize, we find the equation of the tangent line.

$$f'(x) = \frac{1}{2}(1 - x)^{-1/2}(-1)$$

so $f'(0) = -\frac{1}{2}$, and the point is $(0, 1)$.

$$y - 1 = -\frac{1}{2}x, \quad \text{or} \quad y = -\frac{1}{2} + 1$$

35. Find $f'(x)$ directly from the definition of the derivative (using limits):

Recall that:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

(a) $f(x) = \sqrt{3 - 5x}$

$$\lim_{h \to 0} \frac{\sqrt{3 - 5x - 5h} - \sqrt{3 - 5x}}{h} \cdot \frac{\sqrt{3 - 5x - 5h} + \sqrt{3 - 5x}}{\sqrt{3 - 5x - 5h} + \sqrt{3 - 5x}}$$

$$\lim_{h \to 0} \frac{3 - 5x - 5h - 3 + 5x}{h\left(\sqrt{3 - 5x - 5h} + \sqrt{3 - 5x}\right)}$$

$$\frac{-5}{2\sqrt{3 - 5x}}$$

(b) $f(x) = x^2$

$$\lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} =$$

$$\lim_{h \to 0} \frac{h(2x + h)}{h} = 2x$$

36. True/False: The equation of the tangent line to $f(x) = e^x - e^{-2x}$ at $x = 0$ is:

$$y - 0 = (e^x + 2e^{-2x})(x - 0)$$

False. Although $f'(x) = \mathrm{e}^x + 2\mathrm{e}^{-2x}$, we want $f'(0) = 3$.

37. Differentiate:

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{x} & \text{if } x < 0 \end{cases}$$

Correction: $-\sqrt{x} = -\sqrt{-x}$

Is f differentiable at $x = 0$? Explain.

f will not be differentiable at $x = 0$. Note that, if $x > 0$, then $f'(x) = \frac{1}{2\sqrt{x}}$, so $f'(x) \to \infty$ as $x \to 0^+$

38. $f(x) = |\ln(x)|$. Find $f'(x)$. Recall that, if $f(x) = |x|$, then $f'(x) = \frac{x}{|x|}$, so:

$$f'(x) = \frac{\ln(x)}{|\ln(x)|} \cdot \frac{1}{x}$$

39. $f(x) = xe^{g(\sqrt{x})}$. Find $f'(x)$.

$$f'(x) = e^{g(\sqrt{x})} + xe^{g(\sqrt{x})}g'(\sqrt{x})\frac{1}{2\sqrt{x}}$$

40. Let $f(3) = 2$, and $f'(3) = -1$. If g is the inverse of f, (a) At what ordered pair can we evaluate the derivative of g? (b) Compute that derivative.

For part (a), if $f(3) = 2$, then $g(2) = 3$ (Because g is the inverse of f). Now, since $g(f(x)) = x$,

$$g'(f(x))f'(x) = 1 \quad \Rightarrow \quad g'(f(x)) = \frac{1}{f'(x)}$$

so that $g'(2) = g'(f(3)) = \frac{1}{f'(3)} = -1$

41. Find a formula for dy/dx: $x^2 + xy + y^3 = 0$.

$$2x + y + xy' + 3y^2y' = 0 \Rightarrow y'(x + 3y^2) = -2x - y$$

$$y' = \frac{-2x - y}{x + 3y^2}$$

42. Show that 5 is a critical number of $g(x) = 2 + (x - 5)^3$, but that g does not have a local extremum there.

$g'(x) = 3(x - 5)^2$, so $g'(5) = 0$.

By looking at the sign of $g'(x)$ (First derivative test), we see that $g'(x)$ is always non-negative, so g does not have a local min or max at $x = 5$.

43. Find the slope of the tangent line to the following at the point (3,4): $x^2 + \sqrt{y}x + y^2 = 19$

Correction: $x^2 + \sqrt{y}x + y^2 = 31$

$$2x + \frac{1}{2}y^{-1/2}y'x + \sqrt{y} + 2yy' = 0$$

At $x = 3, y = 4$:

$$6 + \frac{3}{4}y' + 2 + 8y' = 0 \quad \Rightarrow \quad y' = \frac{-32}{35}$$

$$y - 4 = \frac{-32}{35}(x - 3)$$

44. Find the critical values: $f(x) = |x^2 - x|$

    One way to approach this problem is to look at it piecewise. Use a table to find where $f(x) = x(x-1)$ is positive or negative:

    $$f(x) = \begin{cases} x^2 - x & \text{if } x \le 0, \text{ or } x \ge 1 \\ -x^2 + x & \text{if } 0 < x < 1 \end{cases}$$

    So $f'(x) = 0$ if $2x - 1 = 0$, or $x = \frac{1}{2}$.

    $f'(x)$ does not exist at $x = 0$ and $x = 1$. (Draw the graph for a quick check). We can also see this:

    $$f(x) = \begin{cases} 2x - 1 & \text{if } x < 0, \text{ or } x > 1 \\ -2x + 1 & \text{if } 0 < x < 1 \end{cases}$$

    At $x = 0$, from the left, $f'(x) \to 1$ and from the right, $f'(x) \to -1$.

    At $x = 1$, from the left, $f'(x) \to -1$, and from the right, $f'(x) \to 1$.

45. Does there exist a function f so that $f(0) = -1$, $f(2) = 4$, and $f'(x) \le 2$ for all x?

    We check the Mean Value Theorem:

    $$f'(x) = \frac{f(2) - f(0)}{2 - 0} = \frac{5}{2}$$

    Since $\frac{5}{2} > 2$, there can exist no function like that (that is continuous).

46. Milk is being produced in a spherical cow so that it's volume increases at a rate of 100 cm^3/s. How fast is the radius of the cow increasing when its diameter is 50 cm?

    The volume of the cow:

    $$V = \frac{4}{3}\pi r^3$$

    (I would give such a formula on the exam) The question asks us to determine $\frac{dr}{dt}$ when $r = 25$ and $\frac{dV}{dt} = 100$.

Differentiating:

$$\frac{dV}{dt} = 4\pi r^2 \, \frac{dr}{dt}$$
$$100 = 4\pi(25)^2 \, \frac{dr}{dt}$$

so $\frac{dr}{dt} = \frac{1}{25\pi}$

Side note: A very old math joke had the punchline: "Consider a spherical cow..." You might guess what the joke was.

47. Linearize $f(x) = \sqrt{1+x}$ at $x = 0$.

    Point: $x = 0, y = 1$

    Slope: $f'(0) = \frac{1}{2}$

    Line: $y - 1 = \frac{1}{2}(x - 0)$, or $y = \frac{1}{2}x + 1$

48. Find dy is $y = \sqrt{1-x}$ and evaluate dy if $x = 0$ and $dx = 0.02$.

$$dy = \frac{-1}{2\sqrt{1-x}}\, dx, \quad \Rightarrow dy = \frac{1}{2\sqrt{1-0}} \cdot 0.02 = 0.01$$

49. Fill in the question marks: If f'' is positive on an interval, then f' is (increasing) and f is (concave up).

50. If $f(x) = x - \cos(x)$, x is in $[0, 2\pi]$, then find the value(s) of x for which

    (a) $f(x)$ is greatest and least. Candidates are at $x = 0, x = 2\pi$ and $1 + \sin(x) = 0$ (which is $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$).

Value of x	Value of y
0	-1
$\pi/2$	$\pi/2 \approx 1.57$
$3\pi/2$	$3\pi/2 \approx 4.71$
2π	$2\pi - 1 \approx 5.28$

    so the maximum occurs at $x = 2\pi$, and the minimum is at $x = 0$.

    (b) $f(x)$ is increasing most rapidly.

    f is increasing most rapidly where the derivative is the largest. We look for the largest value of $1 + \sin(x)$, which occurs at $x = \pi/2$ (draw a quick sketch).

    (c) The slopes of the lines tangent to the graph of f are increasing most rapidly.

    Where is f' increasing most rapidly? Where f'' has a maximum. Since $f''(x) = \cos(x)$, it has a maximum at $x = 0$ and $x = 2\pi$.

51. Show there is *exactly* one root to: $\ln(x) = 3 - x$ between 2 and e, then use Newton's Method to approximate it (accurate to 3 decimal places). (we won't be doing Newton's Method on the exam, so only the first part is done below):

    Use the Intermediate Value Theorem, using $f(x) = \ln(x) - 3 + x$ We'll show that $f(x) = 0$ between 2 and e.

    At $x = 2$, $f(2) = \ln(2) - 3 + 2 \approx -0.3068$. At $x = e$, $f(e) = \ln(e) - 3 + e \approx 0.71828$. Since $f(2) < 0$ and $f(e) > 0$, (and f is continuous on $[2, e]$), then $f(x) = 0$ for some x in the interval $(2, e)$.

52. Use differentials to find a formula for the approximate volume of a thin cylindrical can with height h, inner radius r, and thickness Δr.

    For a can, the volume will be $V = \pi r^2 h$. We should assume that h is constant, and r is the only thing changing, so that

    $$V(r) = \pi r^2 h \quad dV = 2\pi r h \; dr$$

    Setting $dr = \Delta r$, we get that:

    $$dV = 2\pi r h \Delta r$$

    and dV is the approximate change in Volume.

53. Sketch the graph of a function that satisfies all of the given conditions:

    (a) $f(1) = 5, f(4) = 2$
    (b) $f'(1) = f'(4) = 0$,
    (c) $\lim_{x\to 2^+} f(x) = \infty$
    (d) $\lim_{x\to 2^-} f(x) = 3$, and $f(2) = 4$

54. Find the domain of $f(x) = \log_3(x^4 - 8x^3 + 15x^2)$

    For the log, the input must be non-negative, so we need to have $x^4 - 8x^3 + 15x^2 > 0$. Factor and use a table to see that $x > 5$ or $x < 3$.

55. Find the dimensions of the rectangle of largest area that has its base on the x-axis and the other two vertices above the x-axis on the parabola $y = 8 - x^2$.

    If we let x be the length from the origin to the (positive) side of the rectangle, then the lengths of the rectangle are $2x$ and $8 - x^2$. The area is then:

    $$A(x) = 2x(8 - x^2), \quad x \in [0, \sqrt{8}]$$

    The area at the endpoints is zero, so we check the derivative:

    $$A'(x) = 16 - 6x^2$$

    For $A'(x) = 0$, $x = \sqrt{\frac{8}{3}}$. The dimensions of the rectangle are therefore $2\sqrt{\frac{8}{3}}$, $\frac{16}{3}$.

56. Use Newton's Method to approximate x_1, x_2, x_3, for $x_0 = 0.5$ and $f(x) = 2x - \cos(x)$.

Won't be on the Final.