Review Solutions: Chapter 11

1. What does it mean to say that a series converges?

   SOLUTION: We define the n^{th} partial sum s_n as follows:

   $$S_1 = a_1 \qquad S_2 = a_1 + a_2 \qquad S_3 = a_1 + a_2 + a_3 \quad \cdots \quad S_n = \sum_{k=1}^{n} a_n$$

   The partial sums S_n form a sequence (of numbers). The (infinite) series is said to *converge* to a sum S if and only if the limit of the partial sums is S. That is, if the limit

   $$\lim_{n\to\infty} S_n = S$$

   then the series is said to converge, and the sum is said to be equal to S:

   $$\sum_{n=1}^{\infty} a_n = S$$

2. Does the given sequence or series converge or diverge?

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$

   SOLUTION: Using the dominating terms, this looks a lot like $\sum \frac{1}{n}$, so we use the limit comparison (note that both series, the one given and the template, have all positive terms)

   $$\lim_{n\to\infty} \frac{\frac{1}{n-\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n-\sqrt{n}} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{\sqrt{n}}} = 1$$

   Therefore, both series diverge together (by the limit comparison test).

   (b) Take the limit:

   $$\lim_{n\to\infty} \ln\left(\frac{n+2}{n}\right) = \ln\left(\lim_{n\to\infty} \frac{n+2}{n}\right) = \ln(1) = 0$$

   (c) $\left\{\frac{n}{1+\sqrt{n}}\right\}$

   SOLUTION: Take the limit; You can use L'Hospital's rule if you like. To be precise, we ought to change notation to x (since you cannot formally take the derivative of a discrete sequence):

   $$\lim_{x\to\infty} \frac{x}{1+\sqrt{x}} = \lim_{x\to\infty} \frac{1}{1/2\sqrt{x}} = \lim_{x\to\infty} 2\sqrt{x} = \infty$$

   Therefore, the sequence diverges.

   (d) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+1}{n^3-1}$

   SOLUTION: This looks again like the harmonic series (which diverges). Use the limit comparison with $\sum 1/n$:

   $$\lim_{n\to\infty} \frac{\frac{n^2-1}{n^3-1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3-n}{n^3-1} = 1$$

   Therefore, both series diverge together by the limit comparison test.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{5-2\sqrt{n}}{n^3}$

We can temporarily break this apart to see if the pieces converge:

$$\sum_{n=1}^{\infty} \frac{5-2\sqrt{n}}{n^3} = \sum_{n=1}^{\infty} \frac{5}{n^3} - 2\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3}$$

Both of these are p−series, the first with $p = 3$, the second with $p = \frac{5}{2}$, therefore they converge separately, and so the sum also converges.

A good thing to know: The sum of two convergent series is a convergent series.

(f) $\displaystyle\sum_{n=1}^{\infty} (-6)^{n-1} 5^{1-n}$

SOLUTION: First, let's rewrite the terms of the sum:

$$(-6)^{n-1} 5^{1-n} = \frac{(-6)^{n-1}}{5^{n-1}} = \left(\frac{-6}{5}\right)^{n-1}$$

so that this is a geometric series with $r = \frac{-6}{5}$. Since $|r| > 1$, this series diverges.

(g) $\left\{\frac{n!}{(n+2)!}\right\}$

SOLUTION: We first simplify:

$$\frac{n!}{(n+2)!} = \frac{1}{(n+1)(n+2)}$$

so the limit as $n \to \infty$ is 0.

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{5^n n!}$

SOLUTION: With factorials and powers, use the Ratio Test. Because all terms are always positive, we can drop the absolute value signs (if it converges, it would be absolute convergence). Before taking the limit, we can simplify algebraically:

$$\frac{a_{n+1}}{a_n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{5^{n+1}(n+1)!} \cdot \frac{5^n n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{2n+1}{5(n+1)} = \frac{2n+1}{5n+5}$$

Now, take the limit:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{5+\frac{5}{n}} = \frac{2}{5}$$

Since the limit is less than 1, the series converges (absolutely) by the Ratio Test.

(i) $\displaystyle\sum_{n=2}^{\infty} \frac{3^n + 2^n}{6^n}$

A sum of (convergent) geometric series is also convergent. In fact, we can find the sum to which the series will converge:

$$\sum_{n=2}^{\infty} \frac{3^n + 2^n}{6^n} = \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n = \frac{(1/2)^2}{1-(1/2)} + \frac{(1/3)^2}{1-(1/3)} = \frac{2}{3}$$

(j) $\left\{\sin\left(\frac{n\pi}{2}\right)\right\}$
SOLUTION: Write out the first few terms of the sequence:

$$1, 0, -1, 0, 1, 0, -1, \ldots$$

so the sequence diverges.

(k) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$
SOLUTION: We see the terms go to zero like $\frac{1}{n^3}$ (that would be a convergent p series). Therefore, use the limit comparison test:

$$\lim_{n\to\infty} \frac{n^3}{n(n+1)(n+2)} = 1$$

so the series converges by the limit comparison test.

NOTE: Did you try to use the Ratio Test? The Ratio (and Root) tests always give an inconclusive answer for any $p-$series.

(l) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n\sqrt{n}}$
SOLUTION: First, do the terms go to zero? The maximum value of the sine function is 1, and all terms of the sum are positive, so:

$$\frac{\sin^2(n)}{n^{3/2}} \leq \frac{1}{n^{3/2}}$$

so the terms do go to zero. Actually, we've also done a direct comparison with the $p-$series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which converges.

(m) $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{n^2 9^n}$
SOLUTION: Ratio Test (note that the negative sign is meaningless since $(-1)^{2n} = 1$). Start with some algebra to simplify before taking the limit:

$$\frac{5^{2n+2}}{(n+1)^2 9^{n+1}} \cdot \frac{n^2 9^n}{5^{2n}} = \left(\frac{n}{n+1}\right)^2 \cdot \frac{5^{2n}5^2}{5^{2n}} \cdot \frac{9^n}{9^{n+1}} = \left(\frac{n}{n+1}\right)^2 \cdot \frac{25}{9}$$

The limit as $n \to \infty$ is $25/9$:

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 \cdot \frac{25}{9} = \left(\lim_{n\to\infty} \frac{n}{n+1}\right)^2 \cdot \frac{25}{9} = \frac{25}{9} > 1$$

Therefore, the series diverges by the Ratio Test.

(n) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)2^n}$
SOLUTION: It looks like the terms are going to zero like $\frac{1}{2^n}$, so let's compare it to $\sum(1/2)^n$, which is a convergent geometric series.

$$\frac{n}{n+1} \cdot \frac{1}{2^n} \leq \frac{1}{2^n}$$

So the series converges by a direct comparison. The Ratio Test would work well here, too.

3. Find the sum of the series

(a) $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{2^{2n}}$

SOLUTION: Do some algebra first to make it look more like a geometric series:

$$\frac{(-3)^{n-1}}{2^{2n}} = \frac{(-3)^n(-3)^{-1}}{(2^2)^n} = -\frac{1}{3} \cdot \left(-\frac{3}{4}\right)^n$$

Now, this is a convergent series with $a = -1/3$ and $r = -3/4$. The sum is:

$$\frac{(-1/3)(-3/4)}{1 + \frac{3}{4}} = \frac{1}{4} \cdot \frac{4}{7} = \frac{1}{7}$$

(b) $\sum_{n=2}^{\infty} \frac{(x-3)^{2n}}{3^n}$

This is a geometric series with $r = \frac{(x-3)^2}{3}$. Putting it into the formula for the sum,

$$\frac{\left(\frac{(x-3)^2}{3}\right)^2}{1 - \frac{(x-3)^2}{3}} = \frac{(x-3)^4}{9} \cdot \frac{3}{3-(x-3)^2} = \frac{3(x-3)^4}{3-(x-3)^2}$$

(c) A telescoping series is one where the middle terms all cancel out. Writing out the terms of the sum, we see that:

$$\sum_{n=1}^{\infty} \frac{4}{n+4} - \frac{4}{n+5} = \left(\frac{4}{5} - \frac{4}{6}\right) + \left(\frac{4}{6} - \frac{4}{7}\right) + \left(\frac{4}{7} - \frac{4}{8}\right) + \cdots$$

All the terms cancel except for the first, so the sum is 4/5.

4. (a) $\sum \frac{n!x^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

SOLUTION:

Use the Ratio Test (and remember to use the absolute value signs!). First a little algebra:

$$\frac{(n+1)!|x|^{n+1}}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!|x|^n} = \frac{n+1}{2n+1}|x|$$

Now take the limit and apply the Ratio test:

$$|x| \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{|x|}{2} < 1 \quad \Rightarrow \quad |x| < 2$$

Therefore, the radius of convergence is 2.

(b) $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n^2 5^n}$

SOLUTION: Use the Ratio test- First simplify.

$$\frac{|x|^{n+1}}{(n+1)^2 5^{n+1}} \cdot \frac{n^2 5^n}{|x|^n} = \left(\frac{n}{n+1}\right)^2 \cdot \frac{|x|}{5}$$

Now take the limit and apply the test:

$$\frac{|x|}{5} \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = \frac{|x|}{5} \left(\lim_{n\to\infty} \frac{n}{n+1}\right)^2 = \frac{|x|}{5} < 1 \quad \Rightarrow \quad |x| < 5$$

The radius of convergence is 5. When we test $x = -5$ and $x = 5$, we get convergent p series ($\sum 1/n^2$ and $\sum(-1)^n/n^2$, respectively. Therefore, the interval of convergence is

$$[-5, 5]$$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n(x-3)}{\sqrt{n+3}}$

SOLUTION: Another Ratio Test... In this case, the series is centered at $x = 3$, so we'll have an exciting change of pace in calculating the interval of convergence! Here we go- As usual, do the algebra first:

$$\frac{2^{n+1}|x-3|^{n+1}}{\sqrt{n+4}} \frac{\sqrt{n+3}}{2^n|x-3|^n} = 2|x-3|\sqrt{\frac{n+3}{n+4}}$$

The limit can be brought under the radical sign since the square root is a continuous function:

$$2|x-3| \lim_{n\to\infty} \sqrt{\frac{n+3}{n+4}} = 2|x-3|\sqrt{\lim_{n\to\infty} \frac{n+3}{n+4}} = 2|x-3|$$

To apply the Ratio test, if $2|x-3| < 1$, the series will converge absolutely. Therefore, the radius of convergence is 1/2 and to find the interval of convergence, we test the endpoints:

$$-\frac{1}{2} < x-3 < \frac{1}{2} \quad \Rightarrow \quad \frac{5}{2} < x < \frac{7}{2}$$

If we put in $x = 5/2$, the series becomes

$$\sum \frac{2^n \cdot \left(\frac{-1}{2}\right)^n}{\sqrt{n+3}} = \sum \frac{(-1)^n}{\sqrt{n+3}}$$

This will converge by the Alternating Series Test (diverges absolutely since it is similar to a divergent p-series): (i) It is alternating. (ii) It is decreasing: $\sqrt{n+4} > \sqrt{n+3}$, so $1/\sqrt{n+4} < 1/\sqrt{n+3}$. (iii) The terms go to zero.

If we put in $x = 7/2$, we get something similar to a divergent p series, which diverges:

$$\sum \frac{1}{\sqrt{n+3}}$$

We could show it by the limit comparison test with $1/\sqrt{n}$.

Summary: The interval is $[5/2, 7/2)$

5. Find a series for each of the following.

   (a) $\dfrac{1}{1-3x}$
   SOLUTION: This looks kinda like the sum of a geo series:
   $$\frac{1}{1-r} = \frac{1}{1-3x} \qquad \Rightarrow \qquad r = 3x$$
   The series is given by: $\displaystyle\sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} 3^n x^n$. It converges for $|x| < 1/3$.

   (b) $\dfrac{x^2}{1+x}$
   SOLUTION: This looks kinda like the sum of a geo series:
   $$\frac{x^2}{1+x} = x^2 \cdot \frac{1}{1+x} = x^2 \cdot \frac{1}{1-(-x)} = x^2 \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n x^{n+2}$$
   It converges when $|x| < 1$.

6. True or False, and give a short reason:

   (a) If $\lim_{n\to\infty} a_n = 0$, then the series $\sum a_n$ is convergent.
   FALSE. For example, $1/n$ goes to zero, but the series diverges.

   (b) If $\sum c_n 6^n$ is convergent, so is $\sum c_n(-2)^n$.
   TRUE. This is an interesting question, since we don't know the base point- but we can take the base point to be any value!
   For the convergent series, $x - a = 6$, so $x = 6 + a$. Therefore, x could be as much as 6 units from the base point for convergence. To get the quantity -2, we would have:
   $$x - a = -2 \qquad \Rightarrow \qquad x = -2 + a$$
   and so in this case, x is only 2 units away from the base point- Therefore, it must converge in this case as well.

   (c) The Ratio Test can be used to determine if a $p-$series is convergent.
   FALSE: Unfortunately, the Ratio Test always fails for $p-$series. For example, given $1/n^p$, the Ratio Test gives:
   $$\lim_{n\to\infty} \frac{n^p}{(n+1)^p} = \left(\lim_{n\to\infty} \frac{n}{n+1}\right)^p = 1^p = 1$$

   (d) If $0 \le a_n \le b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.
   False. If $\sum a_n$ were divergent, we could then conclude that $\sum b_n$ diverges.

   (e) $0.9999999\ldots = 1$
   True: Let $s = 0.9999999\ldots$. Then
   $$\begin{array}{rl} 10s & = 9.999999\ldots \\ -s & = -0.999999\ldots \\ \hline 9s & = 9.000000\ldots \end{array}$$
   Therefore, $s = 1$.

(f) If $a_n > 0$ and $\sum a_n$ converges, then $\sum(-1)^n a_n$ converges.

True. Notice that $|(-1)^n a_n| = a_n$ since $a_n > 0$- Therefore, $\sum(-1)^n a_n$ converges absolutely. If a series converges absolutely, it must also converge.

7. Suppose that $\sum_{n=0}^{\infty} c_n(x-1)^n$ converges when $x = 3$ and diverges when $x = -2$. What can be said about the convergence or divergence of the following?

NOTE: There are several ways of solving this, but from what we're given, we know that the series must converge for all x between -1 and 3, and must diverge for all $x \le -2$, and $x > 4$. We don't know what happens at $x = 4$.

(a) $\sum c_n$

In this case, $x = 2$ is in the interval of convergence.

(b) $\sum(-1)^n c_n$

In this case, $x = 0$, which is also in the interval where we know the series converges.

(c) $\sum c_n 3^n$

This point is for $x = 4$. At this value, the series could converge or diverge- We would need more information.

8. Let $a_n = \frac{2n}{3n+1}$

(a) Determine whether $\{a_n\}$ is convergent.

SOLUTION: The *sequence* is convergent, and converges to 2/3.

(b) Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

SOLUTION: Since the sequence of terms a_n went to 2/3, the series diverges (by the Test for Divergence).

9. Same as the previous problem, but use $a_n = \dfrac{1+2^n}{3^n}$

(a) Determine whether $\{a_n\}$ is convergent.

SOLUTION: The *sequence* is convergent. Divide numerator and denominator by 3^n, and we get

$$\lim_{n\to\infty} \frac{\frac{1}{3^n} + \left(\frac{2}{3}\right)^n}{1} = 0 + 0 = 0$$

(b) Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

SOLUTION: This is a sum of two convergent geometric series, so the sum will be convergent as well

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n} = \sum_{n=1}^{\infty} \left[\left(\frac{1}{3}\right)^n + \left(\frac{2}{3}\right)^n\right] = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

(We could actually find the sum, but that wasn't part of the question)

10. Compute the sum $1 + r + r^2 + r^3 + \cdots + r^{99}$.

The idea here is to recall how we formulated the sum of a geometric series...

SOLUTION: Start out by defining $S = 1 + r + r^2 + r^3 + \cdots + r^{99}$. Then

$$\begin{array}{rlllllll} S & = 1 & +r & +r^2 & +r^3 & +\cdots & +r^{99} & \\ -rS & = & -r & -r^2 & -r^3 & -\cdots & -r^{99} & -r^{100} \\ \hline (1-r)S & = 1 & & & & & & -r^{100} \end{array} \quad \Rightarrow \quad S = \frac{1 - r^{100}}{1 - r}$$

11. Explain the difference between absolute and conditional convergence. Which is "better" and why?

SOLUTION: Absolute convergence is "better"- If a series is absolutely convergent, it behaves more like a finite sum. For example, if a series converges absolutely, then any re-arrangement of the terms of the sum will converge to the same number. Unfortunately, if a series is only conditionally convergent, it is possible to re-arrange the terms of the sum so that the series will converge to *any* real number.

12. Determine whether each series converges or diverges. In this question, also determine if the series converges absolutely or conditionally.

(a) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{2}n\right)$

SOLUTION: You should recall that $\sin(\pi/2) = 1$, $\sin(\pi) = 0$, $\sin(3\pi/2) = -1$, and so on. Therefore, the sum is:

$$1 + 0 - 1 + 0 + 1 + 0 - 1 + 0 \cdots$$

This does not converge by the Test for Divergence.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

SOLUTION: If we consider absolute convergence, we have: $\sum \frac{1}{\sqrt{n}}$, which is a divergent $p-$series. Since we have an alternating series, we should then see if the original series converges by the Alternating Series Test:

- Does $b_n = \frac{1}{\sqrt{n}} \to 0$? Yes.
- Is b_n a decreasing sequence? Yes, since $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$.

This series is conditionally convergent.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(k)}{k^3}$

SOLUTION: Check for absolute convergence. We also know that $|\cos(k)| < 1$ for any value of k, so we can perform a direct comparison:

$$\frac{|\cos(k)|}{k^3} \leq \frac{1}{k^3}$$

And this comes from a convergent $p-$series, so the original series converges absolutely.

(d) $\frac{1}{2} - \frac{4}{2^3+1} + \frac{9}{3^3+1} - \frac{16}{4^3+1} + \cdots$

SOLUTION: First, we figure out the pattern in the sum. The numerator seems to be n^2 for $n = 1, 2, 3, 4$. To match up the denominator, it would need to be the following (don't forget that the series alternates!)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^3+1}$$

Next, does it converge? This series will behave like $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$, which converges only conditionally.

By taking the limit, we see that $b_n = \frac{n^2}{n^3+1} \to 0$, so we only need to check to see if b_n is decreasing. It may not be obvious that:

$$\frac{n^2}{n^3+1} > \frac{(n+1)^2}{(n+1)^3+1}$$

so we can also differentiate (and show that the derivative is negative for certain values of x):

$$f(x) = \frac{x^2}{x^3+1} \quad \Rightarrow \quad f'(x) = \frac{-x(x^3-2)}{(x^3+1)^2}$$

We see that this expression is negative for $x > 2$, so b_n is decreasing for $n > 2$. Now we can conclude that the series converges conditionally.