Review Solutions: Exam 3

1. Does the given sequence or series converge or diverge?

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$

   SOLUTION: Using the dominating terms, this looks a lot like $\sum \frac{1}{n}$, so we use the limit comparison (note that both series, the one given and the template, have all positive terms)

   $$\lim_{n\to\infty} \frac{\frac{1}{n-\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n-\sqrt{n}} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{\sqrt{n}}} = 1$$

   Therefore, both series diverge together (by the limit comparison test).

   (b) $\left\{\frac{n}{1+\sqrt{n}}\right\}$

   SOLUTION: Take the limit; You can use L'Hospital's rule if you like. To be precise, we ought to change notation to x (since you cannot formally take the derivative of a discrete sequence):

   $$\lim_{x\to\infty} \frac{x}{1+\sqrt{x}} = \lim_{x\to\infty} \frac{1}{1/2\sqrt{x}} = \lim_{x\to\infty} 2\sqrt{x} = \infty$$

   Therefore, the sequence diverges.

   (c) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{n}{n^2+1}$

   SOLUTION: This looks like an alternating harmonic series.

   First, we should show that the series does not converge absolutely. We can do that using the limit comparison test using $\sum 1/n$.

   $$\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$

   Therefore, both series diverge by the limit comparison test.

   Secondly, does the series converge conditionally? We use the Alternating Series Test:

   $$b_n = \frac{n}{n^2+1}$$

   To show the series is decreasing, it might be easiest to look at the derivative (and show its negative):

   $$f(x) = \frac{x}{x^2+1} \qquad \Rightarrow \qquad f'(x) = \frac{1(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

   We see the denominator is always positive, and the numerator is negative if $x > 1$, so b_n will be decreasing for $n > 1$.

   Secondly, do the terms go to zero? Yes (you can use l'Hospital's rule to show that):

   $$\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{2n} = 0$$

   Therefore, the original series converges conditionally by the Alt. Series Test.

   (d) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n+1}\right)$

   The series diverges since the limit of the terms is not zero:

   $$\lim_{n\to\infty} \ln(n/(3n+1)) = \ln(1/3) \neq 0$$

(e) $\displaystyle\sum_{n=1}^{\infty}(-6)^{n-1}5^{1-n}$

SOLUTION: First, let's rewrite the terms of the sum:

$$(-6)^{n-1}5^{1-n} = \frac{(-6)^{n-1}}{5^{n-1}} = \left(\frac{-6}{5}\right)^{n-1}$$

so that this is a geometric series with $r = \frac{-6}{5}$. Since $|r| > 1$, this series diverges.

(f) $\left\{\frac{n!}{(n+2)!}\right\}$

SOLUTION: We first simplify:

$$\frac{n!}{(n+2)!} = \frac{1}{(n+1)(n+2)}$$

so the limit as $n \to \infty$ is 0.

(g) $\displaystyle\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{5^n n!}$

SOLUTION: With factorials and powers, use the Ratio Test. Because all terms are always positive, we can drop the absolute value signs (if it converges, it would be absolute convergence). Before taking the limit, we can simplify algebraically:

$$\frac{a_{n+1}}{a_n} = \frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{5^{n+1}(n+1)!}\cdot\frac{5^n n!}{1\cdot 3\cdot 5\cdots(2n-1)} = \frac{2n+1}{5(n+1)} = \frac{2n+1}{5n+5}$$

Now, take the limit:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{2+\frac{1}{n}}{5+\frac{5}{n}} = \frac{2}{5}$$

Since the limit is less than 1, the series converges (absolutely) by the Ratio Test.

(h) $\displaystyle\sum_{n=2}^{\infty}\frac{3^n+2^n}{6^n}$

A sum of (convergent) geometric series is also convergent. In fact, we can find the sum to which the series will converge:

$$\sum_{n=2}^{\infty}\frac{3^n+2^n}{6^n} = \sum_{n=2}^{\infty}\left(\frac{1}{2}\right)^n + \sum_{n=2}^{\infty}\left(\frac{1}{3}\right)^n = \frac{(1/2)^2}{1-(1/2)} + \frac{(1/3)^2}{1-(1/3)} = \frac{2}{3}$$

(i) $\left\{\sin\left(\frac{n\pi}{2}\right)\right\}$

SOLUTION: Write out the first few terms of the sequence:

$$1, 0, -1, 0, 1, 0, -1, \ldots$$

so the sequence diverges.

(j) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$

SOLUTION: We see the terms go to zero like $\frac{1}{n^3}$ (that would be a convergent p series). Therefore, use the limit comparison test:

$$\lim_{n\to\infty}\frac{n^3}{n(n+1)(n+2)} = 1$$

so the series converges by the limit comparison test.

NOTE: Did you try to use the Ratio Test? The Ratio (and Root) tests always give an inconclusive answer for any $p-$series.

(k) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n\sqrt{n}}$

SOLUTION: First, do the terms go to zero? The maximum value of the sine function is 1, and all terms of the sum are positive, so:

$$\frac{\sin^2(n)}{n^{3/2}} \leq \frac{1}{n^{3/2}}$$

so the terms do go to zero. Actually, we've also done a direct comparison with the $p-$series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which converges.

(l) $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{n^2 9^n}$

SOLUTION: Ratio Test (note that the negative sign is meaningless since $(-1)^{2n} = 1$). Start with some algebra to simplify before taking the limit:

$$\frac{5^{2n+2}}{(n+1)^2 9^{n+1}} \cdot \frac{n^2 9^n}{5^{2n}} = \left(\frac{n}{n+1}\right)^2 \cdot \frac{5^{2n}5^2}{5^{2n}} \cdot \frac{9^n}{9^{n+1}} = \left(\frac{n}{n+1}\right)^2 \cdot \frac{25}{9}$$

The limit as $n \to \infty$ is 25/9:

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 \cdot \frac{25}{9} = \left(\lim_{n\to\infty} \frac{n}{n+1}\right)^2 \cdot \frac{25}{9} = \frac{25}{9} > 1$$

Therefore, the series diverges by the Ratio Test.

(m) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$

SOLUTION: (Note that the series should start at $n = 2$, since $\ln(1) = 0$)

The terms look like they go to zero. It looks like the Integral Test may work out the best.

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln(x)}}\, dx$$

Let $u = \ln(x)$ with $du = (1/x)\, dx$. We'll change the bounds, too: If $x = 2$, then $u = \ln(2)$, and as $x \to \infty$, $u \to \infty$. Then the integral becomes:

$$\int_{\ln(2)}^{\infty} u^{-1/2}\, du = \sqrt{u}\Big|_{\ln(2)}^{\infty}$$

The square root function goes to infinity as u goes to infinity, so the integral diverges. Therefore, the series diverges as well (by the integral test).

2. Evaluate the integral or show it diverges:

(a) $\displaystyle\int_0^1 \frac{x-1}{\sqrt{x}}\, dx = \lim_{T\to 0^+} \int_T^1 \sqrt{x} - \frac{1}{\sqrt{x}}\, dx = \lim_{T\to 0^+} \left(\frac{2}{3}x^{3/2} - 2x^{1/2}\right|_T^1 = \frac{2}{3} - 2 = -\frac{4}{3}$

Therefore, the integral converges (to $-4/3$).

(b) $\displaystyle\int_2^{\infty} \frac{1}{x\ln(x)}\, dx = \int_{\ln(2)}^{\infty} \frac{1}{u}\, du$, where $u = \ln(x)$. Integrating, we get:

$$\ln(u)|_{\ln(2)}^{\infty} \to \infty$$

So the integral diverges.

(c) $\displaystyle\int_0^{\infty} x^3 \mathrm{e}^{-x^4}\, dx$

Let $u = x^4$, then perform u, du substitution as before:

$$\frac{1}{4}\int_0^{\infty} \mathrm{e}^{-u}\, du = -\frac{1}{4}\mathrm{e}^{-u}\Big|_0^{\infty} = -\frac{1}{4}\left(0 - \mathrm{e}^0\right) = \frac{1}{4}.$$

The integral converges (to 1/4).

3. Show that the integral $\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx$ converges or diverges. HINT: Do not try to compute the antiderivative. Be clear as to your justification.
SOLUTION: Since $\sin^2(x) \le 1$ for all x, then

$$\frac{\sin^2(x)}{x^2} \le \frac{1}{x^2} \qquad \text{for all } x$$

and $\int_1^\infty \frac{1}{x^2}\,dx$ converges, then by the comparison test (for integrals), the original integral converges.

4. Find the sum of the series

(a) $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{2^{2n}}$
SOLUTION: Do some algebra first. This should look like a geometric series(?)

$$\frac{(-3)^{n-1}}{2^{2n}} = \frac{(-3)^n(-3)^{-1}}{(2^2)^n} = -\frac{1}{3}\cdot\left(-\frac{3}{4}\right)^n$$

Now, this is a convergent series with $a = -1/3$ and $r = -3/4$. The sum is:

$$\frac{(-1/3)(-3/4)}{1+\frac{3}{4}} = \frac{1}{4}\cdot\frac{4}{7} = \frac{1}{7}$$

(b) $\sum_{n=2}^{\infty} \frac{(x-3)^{2n}}{3^n}$
This is a geometric series with $r = \frac{(x-3)^2}{3}$. Putting it into the formula for the sum,

$$\frac{\left(\frac{(x-3)^2}{3}\right)^2}{1-\frac{(x-3)^2}{3}} = \frac{(x-3)^4}{9}\cdot\frac{3}{3-(x-3)^2} = \frac{3(x-3)^4}{3-(x-3)^2}$$

It's OK to leave it in that form.

5. Find the radius of convergence. For the last two problems, find the interval of convergence.

(a) $\sum \frac{n!x^n}{1\cdot 3\cdot 5\cdots(2n-1)}$
SOLUTION:
Use the Ratio Test (and remember to use the absolute value signs!). First a little algebra:

$$\frac{(n+1)!|x|^{n+1}}{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!|x|^n} = \frac{n+1}{2n+1}|x|$$

Now take the limit and apply the Ratio test:

$$|x|\lim_{n\to\infty}\frac{n+1}{2n+1} = \frac{|x|}{2} < 1 \quad\Rightarrow\quad |x| < 2$$

Therefore, the radius of convergence is 2.

(b) $\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n^2 5^n}$
SOLUTION: Use the Ratio test- First simplify.

$$\frac{|x|^{n+1}}{(n+1)^2 5^{n+1}}\cdot\frac{n^2 5^n}{|x|^n} = \left(\frac{n}{n+1}\right)^2\cdot\frac{|x|}{5}$$

Now take the limit and apply the test:

$$\frac{|x|}{5} \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = \frac{|x|}{5} \left(\lim_{n\to\infty} \frac{n}{n+1}\right)^2 = \frac{|x|}{5} < 1 \quad \Rightarrow \quad |x| < 5$$

The radius of convergence is 5. When we test $x = -5$ and $x = 5$, we get convergent p series ($\sum 1/n^2$ and $\sum(-1)^n/n^2$, respectively. Therefore, the interval of convergence is

$$[-5, 5]$$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n(x-3)}{\sqrt{n+3}}$

SOLUTION: Another Ratio Test... In this case, the series is centered at $x = 3$, so we'll have an exciting change of pace in calculating the interval of convergence! Here we go- As usual, do the algebra first:

$$\frac{2^{n+1}|x-3|^{n+1}}{\sqrt{n+4}} \frac{\sqrt{n+3}}{2^n|x-3|^n} = 2|x-3|\sqrt{\frac{n+3}{n+4}}$$

The limit can be brought under the radical sign since the square root is a continuous function:

$$2|x-3| \lim_{n\to\infty} \sqrt{\frac{n+3}{n+4}} = 2|x-3|\sqrt{\lim_{n\to\infty} \frac{n+3}{n+4}} = 2|x-3|$$

To apply the Ratio test, if $2|x-3| < 1$, the series will converge absolutely. Therefore, the radius of convergence is $1/2$ and to find the interval of convergence, we test the endpoints:

$$-\frac{1}{2} < x - 3 < \frac{1}{2} \quad \Rightarrow \quad \frac{5}{2} < x < \frac{7}{2}$$

If we put in $x = 5/2$, the series becomes

$$\sum \frac{2^n \cdot \left(\frac{-1}{2}\right)^n}{\sqrt{n+3}} = \sum \frac{(-1)^n}{\sqrt{n+3}}$$

This will converge by the Alternating Series Test (diverges absolutely since it is similar to a divergent p-series): (i) It is alternating. (ii) It is decreasing: $\sqrt{n+4} > \sqrt{n+3}$, so $1/\sqrt{n+4} < 1/\sqrt{n+3}$. (iii) The terms go to zero.

If we put in $x = 7/2$, we get something similar to a divergent p series, which diverges:

$$\sum \frac{1}{\sqrt{n+3}}$$

We could show it by the limit comparison test with $1/\sqrt{n}$.

Summary: The interval is $[5/2, 7/2)$

6. Use a known template to find a series for the following:

(a) $\frac{1}{1+x}$

We want to start by recalling that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \text{ if } |x| < 1 \qquad \text{(Geometric series)}$$

Therefore,

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-1)^n x^n, \text{ if } |x| < 1$$

(b) $\frac{x}{3-x^2}$

Starting from the same place, we want to make our expression look like the sum of a geometric series. Factor out $x/3$ from the expression:

$$\frac{x}{3} \cdot \frac{1}{1-\frac{x^2}{3}} = \frac{x}{3} \cdot \sum_{n=0}^{\infty} \left(\frac{x^2}{3}\right)^n = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{3^{n+1}}$$

7. True or False, and give a short reason:

(a) If $\lim_{n\to\infty} a_n = 0$, then the series $\sum a_n$ is convergent.

FALSE. For example, $1/n$ goes to zero, but the series diverges.

(b) If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

SOLUTION: TRUE. This is equivalent to the Test for Divergence, which says that if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

(c) The Ratio Test can be used to determine if a $p-$series is convergent.

SOLUTION: FALSE. Using the Ratio Test on a $p-$series will give a limit of 1. For example, given $\sum 1/n^p$, then

$$\lim_{n\to\infty} \frac{n^p}{(n+1)^p} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p = 1$$

(d) If $0 \le a_n \le b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

SOLUTION: FALSE. If $\sum a_n$ were divergent, we could then conclude that $\sum b_n$ diverges.

(e) If $a_n > 0$ for all n and $\sum a_n$ converges, then $\sum (-1)^n a_n$ converges.

SOLUTION: TRUE. You could re-phrase the question as: If a series converges absolutely, would the corresponding alternating series converge? Yes (absolutely!).

8. Suppose that $\sum_{n=0}^{\infty} c_n(x-1)^n$ converges when $x = 3$ and diverges when $x = -2$. What can be said about the convergence or divergence of the following?

NOTE: There are several ways of solving this, but from what we're given, we know that the series must converge for all x between -1 and 3, and must diverge for all $x \le -2$, and $x > 4$. We don't know what happens at $x = 4$.

(a) $\sum c_n$

In this case, $x = 2$ is in the interval of convergence.

(b) $\sum (-1)^n c_n$

In this case, $x = 0$, which is also in the interval where we know the series converges.

(c) $\sum c_n 3^n$

This point is for $x = 4$. At this value, the series could converge or diverge- We would need more information.

9. To find a power series for $\ln(1+x)$, we use the problem statement:

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}$$

And integrate both sides:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C = \int \frac{1}{1+x}\, dx = \ln(1+x)$$

To find C, we note that, at $x = 0$:

$$C = \ln(1+0) = \ln(1) = 0.$$

So just set $C = 0$.

10. Let $a_n = \dfrac{2n}{3n+1}$

 (a) Determine if $\{a_n\}$ is convergent:

$$\lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3}$$

 Yes, the sequence of terms is convergent.

 (b) Determine whether the series $\sum a_n$ is convergent.

 SOLUTION: The answer is that the series diverges by the divergence test. Our previous answer showed that the terms of the sum converge to $2/3 \neq 0$.

11. Show the series $\sum_{n=1}^{\infty} 1/n^4$ converges by the integral test.

 To check the conditions for the integral test,

$$f(x) = \frac{1}{x^4} \geq 0 \text{ for all } x > 0 \qquad \lim_{x\to\infty} f(x) = 0 \qquad f'(x) = -4x^{-5} < 0 \text{ for all } x > 0$$

 Therefore, f is positive, decreasing and goes to zero. Further,

$$\int_1^{\infty} f(x)\,dx = -x^{-3}\Big|_1^{\infty} = 0 - -1 = 1$$

 Therefore, the corresponding series converges.

12. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

 (a) Prove the series converges by using the Alternating Series Test. Be clear about what you have to check for this.

 In this case, $1/\sqrt{n} > 0$ for all $n > 0$, and

$$n+1 > n \quad \Rightarrow \quad \sqrt{n+1} > \sqrt{n} \quad \Rightarrow \quad \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$$

 And the limit as $n \to \infty$ of $\frac{1}{\sqrt{n}}$ is 0. Therefore, the series converges by the Alt. Series Test.

 (b) If we use 99 terms of the series, what is an estimate of the remainder (of the sum)?

 By our formula for the estimate,

$$R_{99} < \frac{1}{\sqrt{100}} = \frac{1}{10}$$

13. The terms of a series are defined recursively by the equations:

$$a_1 = 2 \qquad\qquad a_{n+1} = \frac{5n+1}{4n+3} a_n$$

 so, for example,

$$a_2 = \frac{6}{7} a_1 = \frac{12}{7}, \qquad a_3 = \frac{11}{11} \cdot a_2 = 1 \cdot \frac{12}{7} = \frac{12}{7}, \cdots$$

 Does the series converge or diverge? (Hint: You have enough information to run a convergence test).

 SOLUTION: Use the Ratio Test. I can leave off the absolute value signs, since the terms will be positive. First, let's do the algebra for the ratio:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{5n+1}{4n+3} a_n}{a_n} = \frac{5n+1}{4n+3}$$

 Therefore, the limit of the ratio is $5/4$, which is larger than 1. The series diverges (by the Ratio Test).