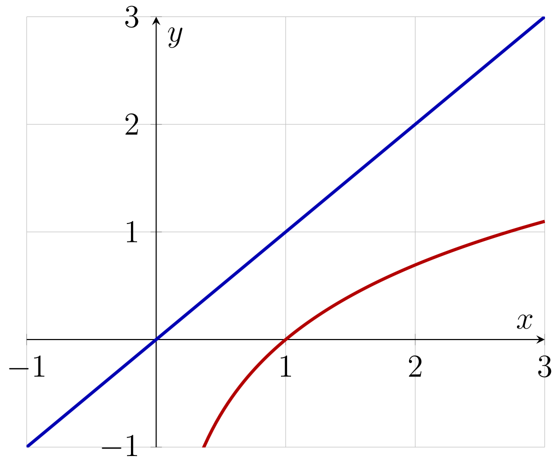
Extra: #209 from Section 5.4

We want to consider the series

$$\sum_{n=1}^{\infty} \frac{\ln(1+1/n)}{n}$$

Graphically, we see that $x > \ln(x)$. Intuitively, $\ln(x)$ grows to infinity, but very slowly (slower than any root function).

Graphs of $y = x$ and $y = \ln(x)$



With that observation, $\ln(x) < x$, or

$$\ln\left(1+\frac{1}{n}\right) \leq 1 + \frac{1}{n} < \frac{1}{n}$$

so that we can use a direct comparison if we multiply both sides by $1/n$:

$$\frac{\ln(1+1/n)}{n} = \frac{1}{n} \cdot \ln\left(1+\frac{1}{n}\right) < \frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n^2}$$

Since $\sum \frac{1}{n^2}$ is a convergent $p-$series, then our series also converges (by the Direct Comparison Test).

We could use the Limit Comparison test with comparing against $1/n^2$, but it's a bit tricky.

$$\lim_{n\to\infty} \frac{\dfrac{\ln(1+1/n)}{n}}{\dfrac{1}{n^2}} = \lim_{n\to\infty} \frac{\ln(1+1/n)}{n} \cdot \frac{n^2}{1} = \lim_{n\to\infty} n \ln(1+1/n)$$

Setting this up to use l'Hospital's rule:

$$\lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(\left(1+\frac{1}{n}\right)\right)}{1/n}$$

Now we can apply l'Hospital's rule to this, noting that $1 + \frac{1}{n} = \frac{n+1}{n}$

$$\lim_{n\to\infty} \frac{\frac{n}{n+1} \cdot (-1/n^2)}{-1/n^2} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

And therefore, the original series converges by the Limit Comparison Test with $\sum \frac{1}{n^2}$