Solutions to Extra Practice in Riemann Sums

1. For each of the following integrals, write the definition using the Riemann sum (and right endpoints), but do not evaluate them:

   (a) $\int_2^5 \sin(3x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(3(2+3i/n)\right)\frac{3}{n}$

   (b) $\int_1^3 \sqrt{1+x}\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+(1+2i/n)}\,\frac{2}{n}$

   (c) $\int_0^2 \mathrm{e}^x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \mathrm{e}^{2i/n}\frac{2}{n}$

2. For each of the following integrals, write the definition using the Riemann sum, and then evaluate them (MUST use the limit of the Riemann sum for credit, and do not re-write them using the properties of the integral):

   (a) $\int_2^5 x^2\,dx$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(2+\frac{3i}{n}\right)^2 \frac{3}{n} = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(4+\frac{12i}{n}+\frac{9i^2}{n^2}\right) =$$

$$\lim_{n\to\infty} \frac{3}{n}\left[\sum_{i=1}^{n} 4 + \sum_{i=1}^{n} \frac{12i}{n} + \sum_{i=1}^{n} \frac{9i^2}{n^2}\right] =$$

$$\lim_{n\to\infty} \frac{3}{n}\left[4\sum_{i=1}^{n} 1 + \frac{12}{n}\sum_{i=1}^{n} i + \frac{9}{n^2}\sum_{i=1}^{n} i^2\right] =$$

$$\lim_{n\to\infty} \frac{3}{n}\left[4n + \frac{12}{n}\frac{n(n+1)}{2} + \frac{9}{n^2}\frac{n(n+1)(2n+1}{6}\right] =$$

$$\lim_{n\to\infty}\left(12 + 18\cdot\frac{n+1}{n} + \frac{9}{2}\cdot\frac{(n+1)(2n+1)}{n^2}\right) = 12+18+9 = 39$$

   (b) $\int_1^3 1-3x\,dx$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[1-3\left(1+\frac{2i}{n}\right)\right]\frac{2}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left[-2-\frac{6i}{n}\right]\frac{2}{n} = \lim_{n\to\infty} -\frac{2}{n}(2n+3(n+1)) = -10$$

   (c) $\int_0^5 1+2x^3\,dx$

   With $(b-a)/n = 5/n$, the Riemann sum is given by:

$$\lim_{n\to\infty} \frac{5}{n}\sum_{i=1}^{n} 1 + 2\frac{5^3 i^3}{n^3} = \lim_{n\to\infty} \frac{5}{n}\left[\sum_{i=1}^{n} 1 + \frac{2\cdot 5^3}{n^3}\sum_{i=1}^{n} i^3\right] =$$

$$\lim_{n\to\infty} \frac{5}{n}\left[n + \frac{2\cdot 5^3}{n^3}\frac{n^2(n+1)^2}{2^2}\right] = \lim_{n\to\infty}\left[5 + \frac{5^4}{2}\frac{(n+1)^2}{n^2}\right] = 5 + \frac{5^4}{2}$$

   Without a calculator, it's fine to leave it in that form.

3. For each of the following Riemann sums, evaluate the limit by first recognizing it as an appropriate integral:

(a) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3}{n}\right) \sqrt{1 + \frac{3i}{n}}$ (Find four different integrals for this one!)
Some options:

$$\int_0^3 \sqrt{1+x}\, dx \qquad \int_1^4 \sqrt{1+(x-1)}\, dx = \int_1^4 \sqrt{x}\, dx \qquad \int_2^5 \sqrt{1+(x-2)}\, dx = \int_2^5 \sqrt{x-1}$$

(b) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + 3 \cdot \frac{25i^2}{n^2}\right) \left(\frac{5}{n}\right)$
Some options:

$$\int_0^5 2 + 3x^2\, dx \qquad \int_1^6 2 + 3(x-1)^2\, dx \qquad \int_2^7 2 + 3(x-2)^2\, dx$$

(c) $\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(3 + \frac{2i}{n}\right) \left(\frac{2}{n}\right)$
Some options:

$$\int_0^2 \sin(3+x)\, dx \qquad \int_3^5 \sin(x)\, dx \text{ etc.}$$