

Review Sheet 1 Solutions

1. Prove by induction: $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

SOLUTION:

- Prove the first case: If $n = 1$, then does $1^2 = \frac{1 \cdot 2 \cdot 3}{6}$? Yes.
- Assume true if $n = k$. That is: $\sum_{i=1}^k i^2 = \frac{k(k+1)(2k+1)}{6}$.

Use the assumption to prove that the statement is true if $n = k + 1$. In that case, we want to show that

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

Here now is the proof:

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^k i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

Get a common denominator and factor out $k + 1$:

$$\begin{aligned} &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{(k+1)(k(2k+1) + 6(k+1))}{6} = \\ &\quad \frac{(k+1)(2k^2 + 7k + 6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6} \end{aligned}$$

2. State the definition of $\int_a^b f(x) dx$.

The definition uses the limit of the Riemann sum. You might also state the definition using right endpoints, as we did below- It may help on other problems.

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \cdot \Delta x_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(a + i \cdot \frac{b-a}{n}\right) \cdot \frac{b-a}{n}$$

KEY IDEA: If the definition is asked for, you must use a Riemann sum, and NOT the Fundamental Theorem of Calculus.

3. True or False, and give a short reason:

- (a) The Alternating Series Test is sufficient to show that a series is conditionally convergent.

FALSE, because if you go directly to the Alternating Series Test, you do not know if the series converges absolutely.

- (b) You can use the Integral Test to show that a series is absolutely convergent.

TRUE. The Direct Comparison, Limit Comparison, Ratio Test and Integral tests were all used on positive series.

(c) Consider $\sum a_n$. If $\lim_{n \rightarrow \infty} a_n = 0$, then the sum is said to converge.

FALSE. For example, $1/n \rightarrow 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Although, if the terms do NOT go to zero, then the sum diverges by the Test for Divergence.

(d) The sequence $a_n = 0.1^n$ converges to $\frac{1}{1-0.1}$

FALSE. This is a sequence, not a series. The sequence converges to zero.

4. Set up an integral for the volume of the solid obtained by rotating the region defined by $y = \sqrt{x-1}$, $y = 0$ and $x = 5$ about the y -axis.

SOLUTION: We might use shells. In that case, the height of a shell is $\sqrt{x-1}$ and the radius is x . The volume is then:

$$\int_1^5 2\pi x \sqrt{x-1} dx$$

ALTERNATE SOLUTION: If we use washers, we would have:

$$\pi \int_0^2 ((y^2 + 1)^2 - 5^2) dy$$

5. Write the area under $y = \sqrt[3]{1+x}$, $1 \leq x \leq 4$ as the limit of a Riemann sum (use right endpoints)

SOLUTION: The i^{th} right endpoint is: $1 + \frac{3i}{n}$. The height at this endpoint is given by:

$$f\left(1 + \frac{3i}{n}\right) = \sqrt[3]{1 + 1 + \frac{3i}{n}} = \sqrt[3]{2 + \frac{3i}{n}}$$

Therefore, the integral is:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt[3]{2 + \frac{3i}{n}} \cdot \frac{3}{n}$$

Secondly, for the arc length, we need to differentiate y :

$$y' = \frac{1}{3}(1+x)^{-2/3} \Rightarrow (y')^2 = \frac{1}{9}(1+x)^{-4/3}$$

so the integral is:

$$\int_1^4 \sqrt{1 + \frac{1}{9(1+x)^{4/3}}} dx$$

6. Find the Taylor series for $f(x) = \sqrt{x}$ centered at $a = 9$.

SOLUTION: The first few derivatives are:

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	\sqrt{x}	2
1	$(1/2)x^{-1/2}$	$(1/4)$
2	$-(1/4)x^{-3/2}$	$-1/32$
3	$(3/8)x^{-5/2}$	$3/256$

(I'll try to make the numbers work out nicely on the exam) Therefore,

$$\sqrt{x} = 2 + \frac{1}{4}(x - 4) - \frac{1}{2 \cdot 32}(x - 4)^2 + \frac{3}{6 \cdot 256}(x - 4)^3 + \dots$$

Main Point: Recall the formula for the Taylor series, and be able to compute the first few terms of a Taylor series.

7. Find $\frac{dy}{dx}$, if $y = \int_{\cos(x)}^{5x} \cos(t^2) dt$

SOLUTION: The general formula we got using the Fundamental Theorem of Calculus, Part I was:

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) dt = f(h(x))h'(x) - f(g(x))g'(x)$$

Therefore, in this particular case,

$$\cos(25x^2) \cdot 5 + \cos(\cos^2(x)) \cdot \sin(x)$$

8. Let $f(x) = e^x$ on the interval $[0, 2]$. (a) Find the average value of f . (b) Find c such that $f_{\text{avg}} = f(c)$.

SOLUTION: Remember the theorem: If f is continuous on $[a, b]$, then there is a c in $[a, b]$ so that:

$$f_{\text{avg}} = f(c) = \frac{1}{b-a} \int_a^b f(x) dx$$

So, we first compute the average, then we'll find the c :

$$\frac{1}{2} \int_0^2 e^x dx = \frac{e^2 - 1}{2}$$

so that $c = \ln\left(\frac{e^2 - 1}{2}\right) \approx 1.16$

9. Use a template series to find the series for $\int \cos(x^2) dx$.

SOLUTION: The template series is the series for $\cos(x)$. In that formula, substitute x^2 for x , then integrate:

$$\cos(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!} \Rightarrow \int \cos(x^2) dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(4n+1)(2n)!} + C$$

10. Does the series converge (absolute or conditional), or diverge?

(a) $\sum_{n=1}^{\infty} \frac{\cos(n/2)}{n^2 + 4n}$ Note that $\cos(n/2) < 1$ for all n . Therefore, by direct comparison, $\left| \frac{\cos(n/2)}{n^2 + 4n} \right| < \frac{1}{n^2}$ Thus, the series converges absolutely by the direct comparison test.

(b) $\sum_{n=1}^{\infty} \frac{n^2 + 1}{5^n}$ Using the ratio test, $\lim_{n \rightarrow \infty} \frac{(n+1)^2 + 1}{5^{n+1}} \cdot \frac{5^n}{n^2 + 1} = \lim_{n \rightarrow \infty} \frac{(n+1)^2 + 1}{n^2 + 1} \cdot \frac{1}{5} = \frac{1}{5}$ The series converges absolutely by the ratio test.

$$(c) \sum_{n=1}^{\infty} \frac{3^n n^2}{n!} \text{ By the ratio test, } \lim_{n \rightarrow \infty} \frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{3}{n+1} = 0$$

11. Find the interval of convergence:

$$(a) \sum_{n=1}^{\infty} \frac{n^2 x^n}{10^n} \text{ (Ratio) } \lim_{n \rightarrow \infty} \frac{(n+1)^2 |x|^{n+1}}{10^{n+1}} \cdot \frac{10^n}{n^2 |x|^n} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{|x|}{10} = \frac{|x|}{10}$$

The radius of convergence is 10. Check the endpoints:

If $x = 10$, the sum is $\sum n^2$, which diverges. If $x = -10$, the sum is $\sum (-1)^n n^2$, which still diverges. The interval of convergence is therefore: $(-10, 10)$

$$(b) \sum_{n=1}^{\infty} \frac{(3x-2)^n}{n 3^n} \text{ (Ratio) } \lim_{n \rightarrow \infty} \frac{|3x-2|^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n 3^n}{|3x-2|^n} = \lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot \frac{|3x-2|}{3} = \left|x - \frac{2}{3}\right| < 1$$

From this, the radius of convergence is 1. The interval so far is: $(-1/3, 5/3)$, so now check the endpoints.

If $x = -\frac{1}{3}$, the sum becomes: $\sum \frac{(-1)^n}{n}$, which converges (conditionally). If $x = \frac{5}{3}$, the sum becomes the Harmonic Series, which diverges.

The interval of convergence is $\left[-\frac{1}{3}, \frac{5}{3}\right)$

$$(c) \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}$$

Be careful with the indices on this one. If the n^{th} term has index $2n-1$, then the $n+1^{\text{st}}$ index is $2(n+1)-1 = 2n+1$. Furthermore,

$$\frac{(2n-1)!}{(2n+1)!} = \frac{1 \cdot 2 \cdot 3 \cdots (2n-2)(2n-1)}{1 \cdot 2 \cdot 3 \cdots (2n-2)(2n-1)(2n)(2n+1)} = \frac{1}{(2n)(2n+1)}$$

Now apply the ratio test:

$$\lim_{n \rightarrow \infty} \frac{|x|^{2n+1}}{(2n+1)!} \cdot \frac{(2n-1)!}{|x|^{2n-1}} = \lim_{n \rightarrow \infty} \frac{x^2}{(2n)(2n+1)} = 0$$

for all x . Therefore, the interval of convergence is the set of all real numbers.

12. Evaluate the integral. You should be able to do these without the table of integrals.

$$(a) \int \frac{1}{y^2 - 4y - 12} dy$$

SOLUTION: Use partial fractions, where

$$\frac{1}{y^2 - 4y - 12} = \frac{1}{(y+2)(y-6)} = \frac{A}{y+2} + \frac{B}{y-6}$$

Then:

$$1 = A(y-6) + B(y+2) \quad \Rightarrow \quad A = -1/8, \quad B = -1/8$$

$$\text{Now integrate: } \int \frac{1}{y^2 - 4y - 12} dy = -\frac{1}{8} \ln |y+2| + \frac{1}{8} \ln |y-6|$$

(b) $\int \frac{2}{3x+1} + \frac{2x+3}{x^2+9} dx$

The first integral is a straight u, du substitution with $u = 3x + 1$. The second integral is similar to the one on the quiz except that we don't have to complete the square- split the integral into two:

$$\int \frac{2x}{x^2+9} dx + \int \frac{3}{x^2+9} dx$$

On the first integral, use u, du substitution with $u = x^2 + 9$, and on the second integral, use either trig substitution (with triangles), or recall the formula for the inverse tangent.

To use trig substitution, we'll use the triangle where $\tan(\theta) = x/3$, so the hypotenuse is $\sqrt{x^2+9}$, $\sec(\theta) = \frac{\sqrt{x^2+9}}{3}$, and $dx = 3 \sec^2(\theta)$. The integral becomes:

$$\int \frac{3}{x^2+9} dx = \int \frac{9 \sec^2(\theta)}{9 \sec^2(\theta)} d\theta = \theta = \tan^{-1}(x/3)$$

Altogether, we get:

$$\int \frac{2}{3x+1} + \frac{2x+3}{x^2+9} dx = \frac{2}{3} \ln|3x+1| + \ln(x^2+9) + \tan^{-1}(x/3) + C$$

(c) $\int x^2 \cos(3x) dx$ Use integration by parts using a table:

$$\left| \begin{array}{c|c|c} + & x^2 & \cos(3x) \\ - & 2x & \frac{1}{3} \sin(3x) \\ + & 2 & -\frac{1}{9} \cos(3x) \\ - & 0 & -\frac{1}{27} \sin(3x) \end{array} \right| \Rightarrow \int x^2 \cos(3x) dx = \frac{1}{3} x^2 \sin(3x) + \frac{2x}{9} \cos(3x) - \frac{2}{27} \sin(3x) + C$$

(d) $\int_{-2}^2 |x-1| dx$ Here, we want to split the integral at $x = 1$:

$$\int_{-2}^2 |x-1| dx = \int_{-2}^1 -x+1 dx + \int_1^2 x-1 dx = 5$$

(e) $\int \frac{dx}{x \ln(x)}$ Use a u, du substitution with $u = \ln(x)$, $du = \frac{1}{x} dx$. This gives:

$$\int \frac{dx}{x \ln(x)} = \int \frac{1}{u} du = \ln|u| = \ln(\ln(x)) + C$$

(f) $\int x\sqrt{x-1} dx$ It's handy to get rid of square roots where possible. Here, try $u = \sqrt{x-1}$ so that $u^2 = x-1$, or $u^2 + 1 = x$. This also gives $2u du = dx$. Now,

$$\int x\sqrt{x-1} dx = \int (u^2+1) \cdot u \cdot 2u \cdot du = \int 2u^4 + 2u^2 du = \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$$

13. The velocity function is $v(t) = 3t - 5$, $0 \leq t \leq 3$ (a) Find the displacement. (b) Find the distance traveled.

SOLUTION: The displacement is

$$\int_0^3 3t - 5 dt = -\frac{3}{2}$$

The distance traveled is the integral of the absolute value of velocity:

$$\int_0^3 |3t - 5| dt$$

Checking $3t - 5$, we see that it is negative if $t < 5/3$, so we break up the integral:

$$\int_0^3 |3t - 5| dt = \int_0^{5/3} 5 - 3t dt + \int_{5/3}^3 3t - 5 dt = \frac{25}{6} + \frac{16}{6} = \frac{41}{6}$$

14. Use a template series to find the Maclaurin series for: $\int \frac{e^x - 1}{x} dx$

SOLUTION: Starting with the series for e^x , we have:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Therefore, subtract one and divide by x to get:

$$\frac{e^x - 1}{x} = 1 + \frac{1}{2}x + \frac{1}{3!}x^2 + \frac{1}{4!}x^3 + \cdots = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$$

Finally, integrate both sides:

$$\int \frac{e^x - 1}{x} dx = C + \sum_{n=1}^{\infty} \frac{x^n}{n! \cdot n}$$

15. Same idea as the previous problem. Start with $1/(1 + t^3)$, which looks like the sum for a geometric series (with $r = (-t^3)$), and:

$$\frac{1}{1 + t^3} = \sum_{n=0}^{\infty} (-1)^n t^{3n}$$

Now we can multiply both sides by t to get the integrand:

$$\frac{t}{1 + t^3} = \sum_{n=0}^{\infty} (-1)^n t^{3n+1}$$

Finally, integrate both sides:

$$\int \frac{t}{1 + t^3} dt = C + \sum_{n=0}^{\infty} (-1)^n \frac{t^{3n+2}}{3n+2}$$

16. Given y below, find the arc length:

$$y = \int_1^x \sqrt{\sqrt{t} - 1} dt \quad \Rightarrow \quad y' = \sqrt{\sqrt{x} - 1} \quad \Rightarrow \quad 1 + (y')^2 = \sqrt{x} - 1 + 1 = \sqrt{x}$$

Therefore,

$$L = \int_1^{16} x^{1/4} dx = \frac{4}{5} x^{5/4} \Big|_1^{16} = \frac{4}{5} (2^5 - 1) = \frac{124}{5}$$

17. Note that we're just rotating the curve $y = 1 - x^2$, and not the entire region between the x -axis and the curve (like we did for a volume of revolution). Therefore, there is only one surface.

$$r = 2 - y = 2 - (1 - x^2) = 1 + x^2$$
$$ds = \sqrt{1 + (f'(x))^2} dx = \sqrt{1 + 4x^2} dx$$

The integral is therefore

$$\int_0^1 2\pi(1 + x^2)\sqrt{1 + 4x^2} dx$$

18. A leaky 10 kg bucket is lifted from the ground to a height of 12 m at a constant speed with a rope that weighs 0.8 kg/m. Initially, the bucket contains 36 kg of water, but the water leaks at a constant rate and finishes draining just as the bucket reaches the 12 m level. Set up the computations you would make to determine how much work is done. You do not have to actually compute the work, just set up the computations and/or integrals needed. You may keep gravity as $g \text{ m/s}^2$, and use the constant σ if you need the density of water (in kg/m^3).

SOLUTION: We compute the work separately for the bucket, the rope, then the water:

- For the bucket, the mass is 10 kg, so the weight (force) is $10g$. The distance is 12 m, so the work for the bucket is $120g$.
- The mass of the rope is 0.8 kg/m, so it actually "weighs" $0.8g \text{ N/m}$. So if we're x meters from the top, the a segment Δx meters has a force $0.8g\Delta x \text{ N}$, and it travels x meters. Add all the segments together:

$$W = \int_0^{12} (0.8g)x dx$$

- For the water, we lose 36 kg over a 12 meter distance, so the amount of water x meters from the top would be: $36 - 3x \text{ kg}$. This is mass, so we need weight- That would be $(36 - 3x)g \text{ N}$ (in this case, we don't need σ - If the water had been given in cubic meters, we would have needed it). This force is constant over the Δx units up, so that the overall work is:

$$W = \int_0^{12} (36 - 3x)g dx$$

(Also see #17 in Section 6.4)