Section 6.4 HW Notes

1. Force is constant, so the work is $360 \times 20 = 7200$. Note that the time it took is irrelevent to the computation, so the work does not change.

3. In this case, the force is given as $5x^{-2}$, so the work, at x_i^*, moving δx units would be $W_i = F_i \times d_i$, or $W_i = 5x_i^{*-2}\Delta x$. Taking the limit,

$$W = \int_1^{10} \frac{5}{x^2}\, dx$$

7. (Not in HW, included for extra practice) The spring problem requires "Hooke's Law", which says that a spring stretched x units past its natural length will have a restorative force of kx. In Exercise 7, the spring is stretched 4 inches, or 1/3 feet (best to stay with feet and lbs). The force was 10 lbs, or

$$k\frac{1}{3} = 10 \quad \Rightarrow \quad k = 30$$

Therefore, for this spring, we have a restorative force of $F(x) = 30x$ (where x is the number of feet stretched past the natural length of the spring).

If we divide the interval from $x = 0$ to $x = 1/2$ into n equal pieces, the work done on the i^{the} subinterval over Δx feet is approximately:

$$W_i = F \cdot d \approx 30x_i^* \Delta x \quad \Rightarrow \quad W = \int_0^{1/2} 30x\, dx$$

8. We see that it takes 25 N of force to keep it stretched 10 cm past natural length, or 0.1 m. That means, using Hooke's Law

$$k(0.1) = 25 \quad \Rightarrow \quad k = 250$$

And, using this, we stretch the string 5 cm or 0.05 m past natural length:

$$W = \int_0^{0.05} 250x\, dx$$

9. (Compare this to 8) It takes 2J of work to stretch the spring 0.12 m past its natural length. That means:

$$\int_0^{0.12} kx\, dx = 2 \quad \Rightarrow \quad 2 = 0.0072\, k \text{ so that } k \approx 277.78$$

Then integrate using that value of k.

12. Let L be the natural length of the spring (which is the unknown). The first piece of information has us going from 10 cm to 12 cm. If a_1, a_2 are the integral bounds, then:

$$a_1 + L = 0.1 \text{ m} \quad \Rightarrow \quad a_1 = 0.1 - L$$

Similarly, the upper bound would be $0.12 - L$, so the integral we have from this:

$$6 = \int_{0.1-L}^{0.12-L} kx\,dx = \frac{k}{2}\left[(0.12-L)^2 - (0.1-L)^2\right] \quad \Rightarrow \quad 6 = \frac{k}{2}(0.0044 - 0.04L)$$

We have two unknowns, but only one equation so far. We need the second piece of information in order to get the second equation. It has a similar set up to the other:

$$10 = \int_{0.12-L}^{0.14-L} kx\,dx = \frac{k}{2}\left[(0.14-L)^2 - (0.12-L)^2\right] \quad \Rightarrow \quad 10 = \frac{k}{2}(0.0052 - 0.04L)$$

To solve this by substitution, we could take, from the first equation,

$$k = \frac{12}{0.0044 - 0.04L}$$

and substitute into the second equation:

$$20 = \frac{12}{0.0044 - 0.04L}(0.0052 - 0.04L) \quad \Rightarrow \quad 0.0880 - 0.80L = 0.0624 - 0.48L$$

Solving for L, we get 0.08 meters, or 8 cm.

13. How much work to pull the rope up the building?

$$\int_0^{50} \frac{1}{2}x\,dx = 625\text{ft-lbs}$$

Notice that if we pull up only half the rope, the amount of work is not exactly half of the overall work- It would be like having half the rope appearing in the problem as a ball weighing $25 \times \frac{1}{2} = 12.5$ pounds. The work to pull this weight up 25 feet is $(12.5 \times 25) = \frac{625}{2}$ (half the work in part (a)). But, we still have the rope:

$$\int_0^{25} \frac{1}{2}x\,dx = \frac{625}{4}$$

Therefore, the work overall is the sum, which we notice is $(3/4) \times 625$- So it takes 75% of the overall work in the first half (which actually makes some sense, since that has the heavier parts to it).

15. A cable that weighs 2 lb/ft is used to lift 800 lbs of coal up a mine shaft 500 ft deep. Find the work done.

SOLUTION: The work can be split up and added- That is, find the work hauling up the rope, and then add that to the work hauling up just the coal.

- The work hauling up the rope: Let x be the number of feet from the top of the shaft. Subdividing the interval $0 \leq x \leq 500$ into n equal subintervals, hauling up the rope at the i^{th} subinterval and x units down, we'll have $500 - x$ feet of rope (at 2 lb/ft) lifted Δx feet. The amount of work done here is then $2(500 - x)\Delta x$. Integrating, we get

$$W_{\text{rope}} = \int_0^{500} 2(500 - x)\, dx = 250,000 \text{ ft-lbs}$$

- The work hauling up the coal is easier to compute- The force doesn't change, so

$$W = F \cdot d = 800 \cdot 500 = 400,000 \text{ ft-lbs}$$

Altogether, the work is $650,000$ ft-lbs.

16. This is the leaky bucket that we looked at in class. We split the problem into three parts: One was to compute the work lifting the bucket, the second is lifting the rope, and the third is the water. The bucket and rope we've discussed, so we look at the water:

We notice that, at time t, the bucket is $x = 2t$ feet above the bottom of the well, as has $40 - 0.2t$ pounds. We want pounds after so many feet, so we can substitute $t = x/2$, or: At x feet above the well, the bucket weighs

$$40 - \frac{1}{5}\frac{x}{2} = 40 - \frac{1}{10}x \text{ lbs}$$

We lift that through Δx units, so that the work at height x_i^*,

$$W_i = F_i \times d_i = 40 - \frac{1}{10}x_i^*\, \Delta x \quad \Rightarrow \quad W = \int_0^{80} 40 - \frac{1}{10}x\, dx$$

17. Another leaky bucket. In this case, we have 36 kg of water, and when the bucket reaches the top, we have none (bummer!). We lost 36 kg of water over 12 meters, so the bucket has mass:

$$36 - \frac{36}{12}x = 36 - 3x \text{ kg}$$

The work for just the water is therefore

$$\int_0^{12} (9.8)(36 - 3x)\, dx = 2116.8 \text{ J}$$

The work for the bucket is $10 \times 9.8 \times 12 = 1176$ J. The work for the rope is

$$\int_0^{12} (9.8)(0.8)(12 - x)\, dx = 564.48 \text{ J}$$

for a sum total of 3857.28 J.

18. The chain density is $\frac{25}{10} = 2.5$ pounds per foot. If we think of the part of the chain x feet below the ceiling, there is $10 - x$ feet of chain being lifted a distance of $x - (10 - x) = 2x - 10$ feet.

Therefore, the work involved is:

$$\int_5^{10} \frac{5}{2}(2x - 10)\,dx = 5\int_5^{10} x - 5\,dx = 62.5 \text{ ft-lbs}$$

21. A rectangular slice of water Δx meters thick and lying x meters above the bottom has width x and volume $8x\Delta x$ cubic meters. It weighs about $9800 \cdot 8x\,\Delta x$ N and must be lifted $5 - x$ meters up. The total work is then:

$$W = \int_0^3 9800 \cdot 8x \cdot (5 - x)\,dx$$

24. (For extra practice) We use similar triangles in this problem to get the volume of water x units up from the bottom. If the large triangle has side lengths $6, 12$, then the smaller triangle of water going up x units must go across $2x$ units.

At x units from the bottom, a slice of water has volume $10 \cdot 2x \cdot \Delta x$ cubic feet, so to get the weight, multiply by the density of water (it would be given as 62.5 lbs per ft^3). That slice must be lifted $6 - x$ feet, so the work overall is

$$\int_0^6 (62.5)(20x)(6 - x)\,dx = 1250\int_0^6 (6x - x^2)\,dx = \cdots = 45,000 \text{ ft-lbs}$$

25. The given setup is translated to the following, where h is the unknown:

$$4.7 \times 10^5 = \int_h^3 (9.8)(10^3)(5 - x) \cdot 8x\,dx$$

Solving, we get that h is approximately 2.