Exercises in Proof by Induction

Here's a summary of what we mean by a "proof by induction":

> **The Induction Principle:** Let $P(n)$ be a statement which depends on $n = 1, 2, 3, \cdots$. Then $P(n)$ is true for all n if:
>
> - $P(1)$ is true (the *base case*).
> - Prove that $P(k)$ is true implies that $P(k+1)$ is true. This is sometimes broken into two steps, but they go together: Assume that $P(k)$ is true, then show that with this assumption, $P(k+1)$ must be true.

Exercises

1. Prove each using induction:

   (a) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

   (b) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

   (c) $\displaystyle\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$

   (d) $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

   (e) $\displaystyle\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

   (f) $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

   (g) $\displaystyle\sum_{i=1}^{n} (2i-1) = n^2$

   (h) $n! > 2^n$ for $n \geq 4$.

   (i) $2^{n+1} > n^2$ for all positive integers.

2. This exercise refers to the Fibonacci sequence:

   $$1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots$$

   The sequence is defined recursively by $f_1 = 1, f_2 = 1$, then $f_{n+1} = f_n + f_{n-1}$ for each $n > 2$. As before, prove each of the following using induction. You might investigate each with several examples before you start.

   (a) $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$

   (b) $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$

   (c) $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$