Solutions for the Proof by Induction Exercises

1. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

   Proof[1]:

   - We first prove that the statement is true if $n = 1$. In this case, statement becomes: $1 = 1(2)/2$, which is true.
   - We assume that the statement is true if $n = k$. That is,

     $$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

   - We show, using our assumption, that the statement must be true when $n = k+1$. That is,

     $$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}.$$

     We do that by starting with the LHS of the equation, then showing that we can get the RHS:

     $$\begin{aligned}
     \sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) && \text{Break apart the sum} \\
     &= \frac{k(k+1)}{2} + (k+1) && \text{By assumption} \\
     &= \frac{k(k+1) + 2(k+1)}{2} \\
     &= \frac{(k+1)(k+2)}{2} && \text{QED}
     \end{aligned}$$

2. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

   Proof:

   - We first prove that the statement is true if $n = 1$. In this case, statement becomes: $1^2 = 1(2)(3)/6$, which is true.
   - We assume that the statement is true if $n = k$. That is,

     $$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}.$$

[1] Q.E.D. is latin: *quod erat demonstrandum*, used to mean "which is what had to be proven" and signifies the end of a proof. Sometimes a box (□) is employed for the same purpose.

- We show, using our assumption, that the statement must be true when $n = k+1$. That is,

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

We do that by starting with the LHS of the equation, then showing that we can get the RHS:

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 && \text{Break apart the sum} \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{By assumption} \\
&= \frac{(k+1)(k(2k+1)+6(k+1))}{6} && \text{Factor out } (k+1) \\
&= \frac{(k+1)(2k^2+7k+6)}{2} && \\
&= \frac{(k+1)(k+2)(2k+3)}{6} && \text{QED}
\end{aligned}$$

3. $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$

Proof:

- We first prove that the statement is true if $n = 1$. In this case, statement becomes: $2^0 = 2^1 - 1 = 1$, which is true.
- We assume that the statement is true if $n = k$. That is,

$$\sum_{i=1}^{k} 2^{i-1} = 2^k - 1.$$

- We show, using our assumption, that the statement must be true when $n = k+1$. That is,

$$\sum_{i=1}^{k+1} 2^{i-1} = 2^{k+1} - 1.$$

We do that by starting with the LHS of the equation, then showing that we can get the RHS:

$$\begin{aligned}
\sum_{i=1}^{k+1} 2^{i-1} &= \sum_{i=1}^{k} 2^{i-1} + 2^k && \text{Break apart the sum} \\
&= 2^k - 1 + 2^k && \text{By assumption}
\end{aligned}$$

$$= 2 \cdot 2^k - 1$$
$$= 2^{k+1} - 1 \qquad \text{QED}$$

4. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

   Proof:

   - We first prove that the statement is true if $n = 1$. In this case, statement becomes: $1^3 = 1^2(2^2)/4$, which is true.
   - We assume that the statement is true if $n = k$. That is,
   $$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}.$$
   - We show, using our assumption, that the statement must be true when $n = k+1$. That is,
   $$\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2(k+2)^2}{4}.$$
   We do that by starting with the LHS of the equation, then showing that we can get the RHS:
   $$\begin{aligned}\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 && \text{Break apart the sum}\\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 && \text{By assumption}\\ &= \frac{(k+1)^2(k^2+4k+4)}{4} && \text{Factor out } (k+1)^2\\ &= \frac{(k+1)^2(k+2)^2}{4} && \text{QED}\end{aligned}$$

5. $\displaystyle\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

   Proof:

   - We first prove that the statement is true if $n = 1$. In this case, statement becomes: $1/2 = 1/2$, which is true.
   - We assume that the statement is true if $n = k$. That is,
   $$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}.$$

- We show, using our assumption, that the statement must be true when $n = k+1$. That is,

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}.$$

We do that by starting with the LHS of the equation, then showing that we can get the RHS:

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} && \text{Break apart the sum} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} && \text{By assumption} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{k+1}{k+2} && \text{QED}
\end{aligned}$$

6. $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

SOLUTION: This is a duplicate question (same as 5). Sorry!

7. $\displaystyle\sum_{i=1}^{n} (2i-1) = n^2$

Proof:

- We first prove that the statement is true if $n = 1$. In this case, statement becomes: $2(1) - 1 = 1^2$, which is true.
- We assume that the statement is true if $n = k$. That is,

$$\sum_{i=1}^{k} (2i-1) = k^2.$$

- We show, using our assumption, that the statement must be true when $n = k+1$. That is,

$$\sum_{i=1}^{k+1} (2i-1) = (k+1)^2.$$

We do that by starting with the LHS of the equation, then showing that we can get the RHS:

$$\sum_{i=1}^{k+1} (2i-1) = \sum_{i=1}^{k} (2i-1) + (2k+1) \qquad \text{Break apart the sum}$$

$$
\begin{aligned}
&= k^2 + 2k + 1 && \text{By assumption} \\
&= (k+1)^2 && \text{QED}
\end{aligned}
$$

8. $n! > 2^n$ for $n \geq 4$.

   Proof:

   - We first prove that the statement is true if $n = 4$. In this case, statement becomes: $4! = 4(3)(2)(1) > (2 \cdot 2)(2 \cdot 2)$, which is true.
   - We assume that the statement is true if $n = k$. That is, $k! > 2^k$.
   - We show, using our assumption, that the statement must be true when $n = k+1$. That is, we need to show that $(k+1)! > 2^{k+1}$.

     We do that by starting with the LHS of the equation, then showing that we can get the RHS:

$$
\begin{aligned}
(k+1)! \quad &= k!(k+1) && \text{Break apart the sum} \\
&> 2^k(k+1) && \text{By assumption} \\
&\geq 2^k \cdot 2 = 2^{k+1} && \text{QED}
\end{aligned}
$$

9. $2^{n+1} > n^2$ for all positive integers.

   Proof:

   - This one is a bit trickier than the others. Although we only need to check it in the case that $n = 1$, let's check it for a few values of n:

n	2^{n+1}	n^2
1	$2^2 = 4$	$1^2 = 1$
2	$2^3 = 8$	$2^2 = 4$
3	$2^4 = 16$	$3^2 = 9$

     So we see that not only is 2^{n+1} larger than n^2, the difference between them is growing. Now let's try the proof.

   - We assume that the statement is true if $n = k$. That is, $2^{k+1} > k^2$.
   - We show, using our assumption, that the statement must be true when $n = k+1$. That is, we need to show that $2^{k+2} > (k+1)^2$.

$$
\begin{aligned}
2^{k+2} \quad &= 2^{k+1} \cdot 2 \\
&> 2k^2 && \text{By assumption}
\end{aligned}
$$

We would be finished if we can assert that $2k^2 > (k+1)^2$, or that $2k^2 > k^2+2k+1$, or equivalently,

$$k^2 - 2k - 1 > 0$$

The critical value here is where we have equality:

$$k^2 - 2k - 1 = 0 \quad \Rightarrow \quad k = 1 \pm \sqrt{2}$$

For $k > 1 + \sqrt{2} \approx 2.14$, our statement is true. In particular, it is true when $k = 3, 4, 5, \cdots$.

What about the smaller values of k? We have shown that the statement is true for those cases by actual computation.

Therefore, the statement is true for all $n > 0$.