Calculus II

For the Exam...

- The exam will be about $1\frac{1}{2}$ times the length of a normal exam, and we have twice the amount of time to take it.
- As a reminder- If you do well on the final, then your lowest exam score will be replaced by the average of it and the final, so try your best!
- No calculators will be allowed, and no notes. However, I will provide the table of trig integrals and the sum formulas for $\sum i^2$ and $\sum i^3$.

The Integral in Theory

- How to write a "proof by induction" (and do a proof by induction for some basic statements).
- The definition of the definite integral.
  - Write an integral from a Riemann sum.
  - Write a Riemann sum from an integral.
- Interpret the integral in terms of geometry (area).
- The Fundamental Theorem of Calculus, Part I.

  This applies to a function f that is continuous on $[a, b]$.
  - Sets $g(x) = \int_a^x f(t)\,dt$ as a differentiable function of x.
  - Says that this function is a particular antiderivative of f, $g(a) = 0$.
  - Be able to differentiate:
  $$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt$$
- The Fundamental Theorem of Calculus, Part II. The main computational tool of Calculus: If F is any antiderivative of the continuous function f,
$$\int_a^b f(x)\,dx = F(b) - F(a)$$
- Understand the difference in notation:
$$\int f(x)\,dx \qquad \int_a^x f(t)\,dt \qquad \int_a^b f(x)\,dx$$
- Understand the difference in notation:
$$\int_a^b \frac{d}{dx} f(x)\,dx \qquad \frac{d}{dx}\int_a^x f(t)\,dt \qquad \frac{d}{dx}\int_a^b f(x)\,dx$$

- The Mean Value Theorem for Integrals. The average value of f is attained at some c in $[a, b]$. That is, if f is continuous on $[a, b]$, then there is a c in the interval so that:

$$f_{\text{avg}} = f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx$$

  Or, the area of the rectangle whose length is $b - a$ and whose height is $f(c)$ is equal to the integral:

$$f(c)(b-a) = \int_a^b f(x)\, dx$$

- The improper integral (Types I and II) is approximated by a definite integral, and is defined by taking the limit. For example,

$$\int_a^{\infty} f(x)\, dx = \lim_{t \to \infty} \int_a^t f(x)\, dx$$

  NOTE: We need to recall techniques for computing a limit. For example, (i) algebraically simplify, (b) divide by x^n for some n, (c) l'Hospital's rule.

The Integral in Practice

We had several methods to evaluate an integral:

- Using geometry (and/or symmetry)
- u, du, or Substitution (Backwards Chain Rule)
- u, dv, or Integration by Parts (be able to use the tabular form of this)
- Partial Fractions. Also, be able to integrate something of the form $\int \frac{ax+b}{x^2+c}\, dx$
- Powers of sine and cosine. In particular, remember the formulas for $\sin^2(x)$ and $\cos^2(x)$. we also did a couple of examples using tangent and secant.
- Trigonometric substitution and the use of reference triangles.

  Note: Even though a table of integrals will be provided, there are some types of integrals we should still be able to do (See the review sheet for Exam 3).
- The table of integrals can be used as well.

Applications of the Integral

- Be able to compute the volume of a solid of revolution using disks, washers and shells. Let w be either x or y, depending on how the functions are defined. Then:

$$\int_a^b \pi R^2\, dw \qquad \pi \int_a^b (R^2 - r^2)\, dw \qquad \int_a^b 2\pi r h\, dw$$

- Be able to compute the arc length of a curve.

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \quad \text{or} \quad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

  Then the arc length is $\int_a^b ds$

- Be able to compute the surface area for something found by rotating a curve around a line. A shorthand for the formula is given by:

$$\int_a^b 2\pi r\, ds$$

- Work: Recall that work is force times distance (when the force is constant), and $F = ma$ (mass $\times$ acceleration). Be sure you understood the following examples: Hooke's Law (with springs). Pull a rope up the side of a building. Pull a leaky bucket out of a well. Pump water out of an object (like a spherical tank). (Specific examples will be given below)

Sequences to Series to Power Series to Taylor Series

Note the evolution of our notation in these sections:

$$\{a_n\}_{n=1}^{\infty}, \quad \sum_{k=1}^{\infty} a_k, \quad \sum_{k=1}^{\infty} c_k(x-a)^k, \quad \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

- Sequences:
  - What is a sequence?
  - Be able to determine if a sequence converges or diverges (Monotonic Sequence Theorem can be used, l'Hospital's rule, divide by an appropriate quantity, etc.)
- Series: $\sum_{n=1}^{\infty} a_n$
  - Template series: Geometric Series (and the formula for the sum of a geometric series), $p-$series, harmonic series, alternating harmonic series.
  - Convergence of the Series:
    * Test for divergence.
    * (For positive series) The direct ($a_n \leq b_n$) and limit comparison ($\lim_{n\to\infty} \frac{a_n}{b_n}$) tests.
    * (For positive series) The integral test, where $f(n) = a_n$- We integrate $f(x)$.
    * (For abs convergence) The Ratio Test and Root Tests. The Ratio Test is by far the most widely used test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

- * Check conditional convergence last: Alternating Series Test.
    (The series has terms with alternating signs, the (abs value of the) terms are decreasing and the limit is zero).

- Power Series: $\sum_{k=1}^{\infty} c_k(x-a)^k$
  - We have one of three choices for convergence. The series converges: (i) Only at $x=a$, (ii) for all x, or (iii) for $|x-a|<R$, and diverges for $|x-a|>R$. We say that R is the radius of convergence.
  - Convergence is usually determined by the Ratio Test. We must check the endpoints of the interval separately (which gives the *interval of convergence*).
  - Be able to get new series from a given series by differentiation or integration.

- Taylor Series: $\sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ or Maclaurin: $\sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$
  - Construct a Taylor series for an *analytic* function f based at $x=a$ (or a Maclaurin series, which is a Taylor series based at $a=0$).
  - Know the template series: e^x, $\sin(x)$, $\cos(x)$, $\frac{1}{1-x}$. Be able to construct other series from these (like $\sin(x^2)$, for example).
  - Find the sum of a series by recognizing it as a familiar Taylor series.
  - Find a series by integrating or differentiating the Taylor (or Maclaurin) series.