Solutions — Exam II Review

Part I. Methods & Setups for Indefinite Integrals

For each integral, we identify a suitable method and give the key substitution(s). Where helpful, an antiderivative is also recorded.

A. $\displaystyle\int \frac{\sqrt{x^2+16}}{x^2}\,dx$

Method: Trig substitution for $\sqrt{x^2+a^2}$.
$x = 4\tan\theta \;\Rightarrow\; \sqrt{x^2+16} = 4\sec\theta,\; dx = 4\sec^2\theta\,d\theta.$

$$\int \frac{\sqrt{x^2+16}}{x^2}\,dx = \int \frac{4\sec(\theta)}{16\tan^2(\theta)}\,4\sec^2(\theta)\,d\theta = \int \frac{\sec^3(\theta)}{\tan^2(\theta)}\,d\theta$$

(You can stop there for this question)

B. $\displaystyle\int x^2\sin(3x)\,dx$

Method: Integration by parts twice.

sign	u	dv
$+$	x^2	$\sin(3x)$
$-$	$2x$	$(-1/3)\cos(3x)$
$+$	2	$(-1/9)\sin(3x)$
$-$	0	$(1/27)\cos(3x)$

$$\boxed{\int x^2\sin(3x)\,dx = -\frac{x^2}{3}\cos(3x) + \frac{2x}{9}\sin(3x) + \frac{2}{27}\cos(3x) + C}.$$

(You could have stopped after identifying the u, dv, but it was finished for extra practice.

C. $\displaystyle\int \frac{x+1}{(x+3)(x-1)}\,dx$

Method: Partial fractions. Find A, B with

$$\frac{x+1}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}.$$

Solve: $x+1 = A(x-1) + B(x+3) \Rightarrow A+B=1,\ -A+3B=1 \Rightarrow A=\frac{1}{2},\ B=\frac{1}{2}$.
Hence
$$\boxed{\int \frac{x+1}{(x+3)(x-1)}\,dx = \tfrac{1}{2}\ln|x+3| + \tfrac{1}{2}\ln|x-1| + C}.$$

D. $\displaystyle\int \cos^2 x\,\sin^3 x\,dx$

Method: Trig integrals with an odd power of sin.
Write $\sin^3 x = (1-\cos^2 x)\sin x$, let $u=\cos x$, $du = -\sin x\,dx$:
$$\int \cos^2 x\,\sin^3 x\,dx = \int \cos^2 x(1-\cos^2 x)\sin x\,dx = -\int u^2(1-u^2)\,du.$$
Then integrate the resulting polynomial and back-substitute.

E. $\displaystyle\int x\ln(x)\,dx$

Method: Integration by parts.

sign	u	dv
$+$	$\ln(x)$	x
$-$	$1/x$	$\frac{1}{2}x^2$

$$\boxed{\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C}.$$

Note: What happens if you do it this way:

sign	u	dv
$+$	x	$\ln(x)$
$-$	1	
$+$	0	

The entries under $\ln(x)$ would be: $\int \ln(x)\,dx$ (which itself requires integration by parts), then we would antidifferentiate again...

F. $\displaystyle\int \sec^4(2x)\,dx$

Method: If we "reserve" $\sec^2(2x)$ for a u, du substitution, then $u=\tan(2x)$, so the actual $du = 2\sec^2(2x)\,dx$.

Now rewrite the remaining $\sec^2(2x) = \tan^2(2x)+1$, so we get
$$\int \sec^4(2x)\,dx = \int(\tan^2(2x)+1)\sec^2(2x)\,dx = \frac{1}{2}\int(u^2+1)\,du = \frac{u^3}{6}+\frac{u}{2}+C$$
$$\boxed{= \frac{\tan^3(2x)}{6} + \frac{\tan(2x)}{2} + C.}$$

G. $\displaystyle\int \frac{2x-3}{x^3+x}\,dx$

Method: Partial fractions with quadratic factor. Factor $x(x^2+1)$:

$$\frac{2x-3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

Solve $2x-3 = A(x^2+1) + (Bx+C)x \Rightarrow$

$$\begin{cases} x^2: & A+B=0 \\ x: & C=2 \\ 1: & A=-3 \end{cases} \Rightarrow A=-3,\ B=3,\ C=2.$$

Thus

$$\int \frac{2x-3}{x^3+x}\,dx = \int \left(\frac{-3}{x} + \frac{3x+2}{x^2+1}\right) dx = \boxed{-3\ln|x| + \tfrac{3}{2}\ln(x^2+1) + 2\arctan x + C}.$$

H. $\displaystyle\int \frac{x^2}{\sqrt{x^3+4}}\,dx$

Method: u-sub with inner power. Let $u = x^3+4$, $du = 3x^2\,dx$:

$$\int \frac{x^2}{\sqrt{x^3+4}}\,dx = \frac{1}{3}\int u^{-1/2}\,du = \boxed{\frac{2}{3}\sqrt{x^3+4} + C}.$$

I. $\displaystyle\int \frac{\sqrt{4-x^2}}{x}\,dx$

Method: Trig substitution for $\sqrt{a^2-x^2}$.
Take $x = 2\sin\theta$ so $\sqrt{4-x^2} = 2\cos\theta$, $dx = 2\cos\theta\,d\theta$:

$$\int \frac{\sqrt{4-x^2}}{x}\,dx = \int \frac{2\cos\theta}{2\sin\theta}\cdot 2\cos\theta\,d\theta = \int \frac{2\cos^2\theta}{\sin\theta}\,d\theta$$

We would stop there for now.

J. $\displaystyle\int \sqrt{x^2-16}\,dx$

Method: Trig subs for $\sqrt{x^2-a^2}$.
Choice: $x = 4\sec\theta$, so that the integral becomes

$$16\left(\int \sec^3(\theta) - \sec(\theta)\,d\theta\right)$$

and then we could use the formula sheet to evaluate these.

For the second part, using a triangle where $\sec(\theta) = x/4$, or $\cos(\theta) = 4/x$, we get

$$\tan(\theta) = \frac{\sqrt{x^2-16}}{4}$$

Part II. Setups (No Evaluation Required)

A. Volume (about the x-axis) for $f(x) = x^2 - 4x + 5$ on $x \in [1, 4]$:
Disks/Washers:

$$\boxed{V = \pi \int_1^4 \left(x^2 - 4x + 5\right)^2 dx}.$$

B. Arc length of $y = 2x^{3/2}$ on $[0, 1]$:
$y' = 3\sqrt{x}$, so

$$\boxed{L = \int_0^1 \sqrt{1 + \left(3\sqrt{x}\right)^2}\, dx = \int_0^1 \sqrt{1 + 9x}\, dx}.$$

C. Volume (about the y-axis) under $f(x) = \frac{1}{x}$ above the x-axis on $[1, 3]$:
Cylindrical shells (about y-axis):

$$\boxed{V = 2\pi \int_1^3 x \cdot \frac{1}{x}\, dx = 2\pi \int_1^3 1\, dx}.$$

D. Volume (about the y-axis) for region R between $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{x}$ on $[1, 4]$:
Cylindrical shells:

$$\boxed{V = 2\pi \int_1^4 x\left(\sqrt{x} - \frac{1}{x}\right) dx}.$$

E. Volume (about the x-axis) for the region between $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{x}$ on $[1, 3]$:
For $x \in (1, 3]$, $\sqrt{x} \geq \frac{1}{x}$, so

$$\boxed{V = \pi \int_1^3 \left[\left(\sqrt{x}\right)^2 - \left(\tfrac{1}{x}\right)^2\right] dx = \pi \int_1^3 \left(x - \frac{1}{x^2}\right) dx}.$$

F. Surface area of the surface generated by revolving $y = 7x$ on $[0, 1]$ about the x-axis:
Use $S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx$ with $f'(x) = 7$:

$$\boxed{S = 2\pi \int_0^1 7x\sqrt{1 + 49}\, dx = 2\pi\sqrt{50} \int_0^1 7x\, dx}.$$