Solutions: Sample Calc 2 Exam Questions

(Remember that you can use the page of formulas to help)

1. Differentiate using FTC 2 with variable limits.

$$\frac{d}{dx}\int_{3x}^{x^2} \frac{\sin t}{1+t^2}\,dt = \frac{\sin(x^2)}{1+x^2}\cdot(2x) - \frac{\sin(3x)}{1+9x^2}\cdot(3).$$

2. Evaluate.

   (a) $\int_0^1 \frac{d}{dx}\left(e^{\tan^{-1}x}\right)dx = e^{\tan^{-1}x}\Big|_0^1 = e^{\tan^{-1}(1)} - e^{\tan^{-1}(0)} = e^{\pi/4} - 1.$
   (You could leave the last computation off for now).

   (b) $\frac{d}{dx}\int_0^1 e^{\tan^{-1}(x)}\,dx = 0.$ (This is the derivative of a number).

   (c) $\frac{d}{dx}\int_0^x e^{\tan^{-1}(t)}\,dt = e^{\tan^{-1}(x)}.$

3. Right-endpoint Riemann sum with $n = 2$ on $[0, 8]$ for $y = g(x)$.

   Here $\Delta x = \frac{8-0}{2} = 4$ and right endpoints are $x_1 = 4$, $x_2 = 8$. From the graph, we see that $g(4) = 3$ and $g(8) = 7$, so those are the two heights: Approximation (area of two rectangles):

$$R_2 = 3\cdot 4 + 7\cdot 4 = 40$$

4. Limit using the hint $\sum_{i=1}^n i = \frac{n(n+1)}{2}$.

$$\lim_{n\to\infty}\sum_{i=1}^n\left(1+\frac{4i}{n}\right)\cdot\frac{1}{n} = \lim_{n\to\infty}\sum_{i=1}^n\left(\frac{1}{n}+\frac{4i}{n^2}\right) = \lim_{n\to\infty}\sum_{i=1}^n\frac{1}{n} + \sum_{i=1}^n\frac{4i}{n^2} =$$

$$\lim_{n\to\infty}\frac{1}{n}\cdot n + \frac{4}{n^2}\sum_{i=1}^n i = \lim_{n\to\infty} 1 + \frac{4}{n^2}\frac{n(n+1)}{2} = 1 + 2 = 3$$

5. Write each integral as a limit (right endpoints).

   (a) $\int_2^5 x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^n\left(2+\frac{3i}{n}\right)^2\frac{3}{n}.$

   (b) $\int_1^3 (1-3x)\,dx = \lim_{n\to\infty}\sum_{i=1}^n\left(1-3\left(1+\frac{2i}{n}\right)\right)\frac{2}{n}.$

6. Mean Value Theorem for Integrals for $f(x) = x^2$ on $[0, 2]$.

   Find c with $f(c) = \frac{1}{2}\int_0^2 x^2\,dx = \frac{1}{2}\cdot\frac{8}{3} = \frac{4}{3}$.
   Thus $c^2 = \frac{4}{3} \implies c = \frac{2}{\sqrt{3}}$ (in $(0, 2)$).

7. True/False (with reason).

   (a) False. The definite integral is *signed* area; when $f < 0$ it subtracts area.

   (b) True. This is the definition of a (regular) Riemann sum.

   (c) True. Odd integrands integrate to 0 over $[-a, a]$ if integrable.

   (d) False. E.g. $f(x) = \sin x$ over $[0, 2\pi]$ integrates to 0 but is not identically 0.

   (e) True. $\int_a^b c\,dx = c(b-a)$.

   (f) False. $f(x) = |x|$ is continuous but not differentiable at 0.

   (g) True. If f is continuous, $F(x) = \int_a^x f(t)dt$ satisfies $F' = f$.

8. Evaluate limits by recognizing definite integrals.

   (a) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3}{n}\right)\sqrt{1 + \frac{3i}{n}} = \int_0^3 \sqrt{1+x}\,dx = \left[\frac{2}{3}(1+x)^{3/2}\right]_0^3 = \frac{14}{3}$.
   (Also $\int_1^4 \sqrt{u}\,du$ via $u = 1 + x$.)

   (b) The term $\frac{25i^2}{n^2}$ suggests $x_i = \frac{5i}{n}$; then

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + 3 \cdot \frac{25i^2}{n^2}\right)\frac{5}{n} = \int_0^5 \left(2 + 3x^2\right) dx = \left[2x + x^3\right]_0^5 = 10 + 125 = 135.$$

9. Evaluate the integrals

   (a) $\int_1^9 \frac{\sqrt{u} - 2u^2}{u}\,du$
   For the integrand,

$$u^{-1}(u^{1/2} - 2u^2) = u^{-1/2} - 2u$$

   Therefore, the integral becomes:

$$\left(2u^{1/2} - u^2\right|_1^9 = (6 - 81) - (2 - 1) = 77$$

   (b) $\int \left(3x + \frac{1}{x} + \sec^2 x\right) dx = \frac{3}{2}x^2 + \ln|x| + \tan x + C$.

   (c) $\int_{-\pi/4}^{\pi/4} \frac{t^4 \tan t}{2 + \cos t}\,dt = 0$ (odd integrand over symmetric limits).

   (d) $\int_0^3 |x - 2|\,dx = \int_0^2 (2 - x)\,dx + \int_2^3 (x - 2)\,dx = \frac{5}{2}$.
   (Draw a sketch and/or break it up at $x = 2$)

(e) $\int \frac{\cos(\ln x)}{x}\,dx = \sin(\ln x) + C$. (Let $u = \ln(x)$).

(f) $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$. (See the formula page)

(g) $\int_{-1}^{2} \frac{1}{x}\,dx$. We cannot use FTC on this ($1/x$ not continuous on $[-1,2]$).

(h) $\int_{-2}^{-1} \frac{1}{x}\,dx = \ln|x|\Big|_{-2}^{-1} = -\ln 2$.

(i) $\int (1+\tan t)\sec^2 t\,dt$

Let $u = 1+\tan(t)$, and $du = \sec^2(t)\,dt$. The integral is then $\int u\,du = \frac{1}{2}u^2 + C$, or

$$\frac{1}{2}(1+\tan(t))^2 + C$$

(j) $\int \frac{x}{\sqrt{1+x}}\,dx$

Let $u = 1+x$, so $du = dx$. The numerator of the integrand is still x, so substitute $x = u-1$ for that. Now the integrand can be written as:

$$u^{-1/2}(u-1) = u^{1/2} - u^{-1/2}$$

and

$$\int u^{1/2} - u^{-1/2}\,du = \frac{2}{3}u^{3/2} - 2u^{1/2} + C$$

which simplifies to $\frac{2}{3}(1+x)^{3/2} - 2(1+x)^{1/2} + C$.

(k) $\int \frac{y-1}{\sqrt{3y^2-6y+4}}\,dy$ Let $u = 3y^2 - 6y + 4$. Then $du = (6y-6)\,dy = 6(y-1)\,dy$, or $\frac{1}{6}du = (y-1)\,dy$. The integral becomes:

$$\frac{1}{6}\int u^{-1/2}\,du = \frac{2}{6}u^{1/2} + C = \frac{1}{3}u^{1/2} + C = \frac{1}{3}\sqrt{3y^2-6y+4} + C$$

10. Differentiate functions defined by integrals.

(a) $F(x) = \int_0^{x^2} \frac{\sqrt{t}}{1+t^2}\,dt \quad \Rightarrow \quad F'(x) = \frac{\sqrt{x^2}}{1+x^4}\cdot(2x)$

(b) $y = \int_{\sqrt{x}}^{3x} \frac{e^t}{t}\,dt \quad \Rightarrow \quad y' = \frac{e^{3x}}{3x}\cdot 3 - \frac{e^{\sqrt{x}}}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}$

11. Riemann sum as integral:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^9 = \int_0^1 x^9\,dx = \frac{1}{10}.$$

12. Given the graph of $f(t)$, evaluate $F(x) = \int_0^x f(t)\,dt$, where $x = 0, 2and6$.

   (a) $F(0) = 0$.

   (b) $F(2)$ is the area of the half-circle of radius 1, which is $\frac{1}{2}\pi r^2 = \frac{\pi}{2}$.

   (c) $F(6)$: The area of the triangle is $\frac{1}{2}bh = \frac{1}{2}4 \cdot 2 = 4$. Subtract this from the half-circle area for the net area:

$$F(6) = \frac{\pi}{2} - 4$$

   (d) The first $x > 0$ with $F(x) = 0$ occurs where the accumulated signed area from 0 cancels to zero. Note that $pi/2 \approx 1.57$, so estimating x gives somewhere between $x = 3$ and $x = 4$

   (e) Note that at $F(2)$, F stops increasing and starts to decrease until $F(6)$, where we start gaining positive area again. Therefore, there is at least a local minimum for F at $x = 6$.

13. Projectile with constant acceleration.

$$a(t) = -9.8\ \mathrm{m/s}^2, \quad v(0) = 60, \quad h(0) = 3.$$

$$v(t) = 60 - 9.8\,t, \qquad h(t) = 3 + 60t - 4.9t^2,$$

   valid until the ball hits the ground (first $t > 0$ with $h(t) = 0$).

14. Fuel consumed: $c(t) = 7 - e^{-t}$ L/hr.
   Amount in first 2 hours:

$$\int_0^2 \left(7 - e^{-t}\right) dt = \left[7t + e^{-t}\right]_0^2 = 14 + e^{-2} - 1 = 13 + e^{-2} \text{ litres.}$$