Solutions for Extra Examples (Section 5.2)

1. Does the series converge or diverge?

   NOTE: When checking, you should also do a quick check to see if the terms of the sum are going to zero, then proceed from there.

   (a) $\sum \frac{k(k+2)}{(k+3)^2}$

   Since the leading term of the numerator is k^2 and the leading term of the denominator is also k^2, we suspect that the terms of the sum do not go to zero. Therefore, we use the "Divergence Test". We can take the limit a couple of different ways:

   i. Method 1: Divide numerator and denominator by k^2

$$\begin{aligned}\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k(k+2)}{(k+3)^2} &= \lim_{k\to\infty} \frac{k^2+2k}{k^2+6k+9} \\ &= \lim_{k\to\infty} \frac{\frac{k^2}{k^2}+\frac{2k}{k^2}}{\frac{k^2}{k^2}+\frac{6k}{k^2}+\frac{9}{k^2}} \\ &= \lim_{k\to\infty} \frac{1+\frac{2}{k}}{1+\frac{6}{k}+\frac{9}{k^2}} \\ &= \frac{1+0}{1+0+0} = 1\end{aligned}$$

   ii. Method 2: Take the limit using l'Hospital's rule (twice):

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k(k+2)}{(k+3)^2} = \lim_{k\to\infty} \frac{k^2+2k}{k^2+6k+9} = \lim_{k\to\infty} \frac{2k+2}{2k+6} = \frac{2}{2} = 1$$

   In either case, $\lim_{k\to\infty} a_k = 1 \neq 0$, so the series **diverges** by the divergence test.

   (b) $\sum_{n=1}^{\infty} \frac{5}{n^2}$ The series can be written as:

$$\sum_{n=1}^{\infty} \frac{5}{n^2} = 5\sum_{n=1}^{\infty} \frac{1}{n^2}$$

   The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a **p-series** with $p = 2$. Since $p = 2 > 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ **converges**. Therefore, the series $\sum_{n=1}^{\infty} \frac{5}{n^2}$ **converges**.

   (c) $\sum_{n=2}^{\infty} \frac{1+3^n}{5^n}$ We can split the series into two geometric series:

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{1+3^n}{5^n} &= \sum_{n=2}^{\infty} \left(\frac{1}{5^n}+\frac{3^n}{5^n}\right) \\ &= \sum_{n=2}^{\infty} \left(\frac{1}{5}\right)^n + \sum_{n=2}^{\infty} \left(\frac{3}{5}\right)^n\end{aligned}$$

The first series is $\displaystyle\sum_{n=2}^{\infty}\left(\frac{1}{5}\right)^n$ with $r = \frac{1}{5} < 1$, so the series converges.

Similarly, $\displaystyle\sum_{n=2}^{\infty}\left(\frac{3}{5}\right)^n$ has $r = \frac{3}{5} < 1$, so the series also converges.

The original series is a sum of two convergent geometric series, so it also converges.

2. What's wrong?

   The error in the argument below is the manipulation of a **divergent series**.

$$
\begin{aligned}
0 &= 0 + 0 + 0 + 0 + \cdots \\
&= (1-1) + (1-1) + (1-1) + \cdots \\
&= 1 - 1 + 1 - 1 + 1 - 1 + \cdots \\
&= 1 + (-1+1) + (-1+1) + (-1+1) + \cdots \\
&= 1 + 0 + 0 + 0 + 0 + \cdots
\end{aligned}
$$

   Therefore, $0 = 1$.

Explanation of the Error

The series $\sum_{n=0}^{\infty}(-1)^n = 1 - 1 + 1 - 1 + \cdots$ is Grandi's Series, which is a **divergent series**. The sequence of its partial sums oscillates between 1 and 0, so the limit of the partial sums does not exist.

The algebraic properties (such as the associative law, which allows for the arbitrary grouping of terms) used in steps 3 and 4 are only valid for **convergent series**. Since step 3 is divergent, regrouping the terms leads to different results (0 in the first calculation, and 1 in the second calculation), demonstrating that the series does not have a unique, well-defined sum in the standard sense.

Here is a nice video about it: `https://www.youtube.com/watch?v=PCu_BNNI5x4`