Solutions to Quiz 10

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n-1}{n^2+n}$

   SOLUTION: The series does not converge absolutely (it would behave like $\sum 1/n$), so we check for conditional convergence by using the alternating series test.

   - Let $b_n = \frac{n-1}{n^2+n}$, so for $n \geq 1$, $b_n \geq 0$.
   - The function is decreasing:

     $$f(x) = \frac{x-1}{x^2+x} \qquad \Rightarrow \qquad f'(x) = \frac{-x^2+2x+1}{x^2(x+1)^2}$$

     The parabola in the numerator opens downward, so it is decreasing past its vertex, at $x = -b/2a$, or $x = -2/2(-1) = 1$. Therefore, for $x > 1$, the expression is decreasing.
   - The limit is zero:

     $$\lim_{x\to\infty}\frac{x-1}{x^2+x} = \lim_{x\to\infty}\frac{1}{2x+1} = 0$$

     (Yes, sometimes we get sloppy and leave it as an expression in n, but to differentiate, it really ought to be in terms of a real number x, versus the integers n).

   By the Alternating Series Test, the series converges (conditionally).

2. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{10^n}{n!}$. Probably easiest to use the Ratio Test. NOTE: Recall that we're taking the absolute value of the terms, so the $(-1)^n$ goes away:

   $$\lim_{n\to\infty}\frac{10^{n+1}}{(n+1)!}\frac{n!}{10^n} = \lim_{n\to\infty}\frac{10}{n+1} = 0$$

   Since the limit is less than 1, the series converges absolutely.

3. $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{2\cdot 5\cdot 8\cdots(3n+2)}$ Use the Ratio Test again:

   $$\lim_{n\to\infty}\frac{(n+1)!}{2\cdot 5\cdot 8\cdots(3n+2)(3n+5)}\frac{2\cdot 5\cdot 8\cdots(3n+2)}{n!} = \lim_{n\to\infty}\frac{n+1}{3n+5} = \frac{1}{3} < 1$$

   Therefore, the series converges absolutely (by the Ratio Test).

4. $\displaystyle\sum_{n=1}^{\infty} n^2 \mathrm{e}^{-n^3}$

Consider $f(x) = x^2 \mathrm{e}^{-x^3}$. The function f is continuous, positive and the derivative is:

$$f'(x) = -x\mathrm{e}^{-x^3}(3x^3 - 2)$$

so, for $x > 2$, f is decreasing. Now we can use the Integral Test for convergence. You might notice that setting $u = x^3$ will give us a nice substitution:

$$\int_1^\infty x^2 \mathrm{e}^{-x^3}\, dx = \frac{1}{3}\int_{1/e}^\infty \mathrm{e}^{-u}\, du$$

Since $\lim_{u\to\infty} \mathrm{e}^{-u} = 0$, the integral will converge.

Therefore, by the Integral Test, the original series will also converge.

5. $\displaystyle\sum_{n=1}^\infty \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$

   This series will converge like $1/n^2$, so we can try the limit comparison:

$$\lim_{n\to\infty}\left(\frac{\sqrt{n^2-1}}{n^3+2n^2+5}\cdot\frac{n^2}{1}\right)\frac{1/n^3}{1/n^3} = \lim_{n\to\infty}\frac{\sqrt{n^2-1}/n}{1+\frac{2}{n}+\frac{5}{n^3}} = \lim_{n\to\infty}\frac{\sqrt{1-\frac{1}{n^2}}}{1+\frac{2}{n}+\frac{5}{n^3}} = 1$$

   Since the limit is in the interval $(0,\infty)$, the two series will converge or diverge together. Since $\sum 1/n^2$ is convergent, they will both converge (absolutely, since the terms are all positive).

6. $\displaystyle\sum_{n=1}^\infty \frac{(2n)^n}{n^{2n}}$. This one is kind of set up for the Root Test:

$$\lim_{n\to\infty}\sqrt[n]{\frac{(2n)^n}{n^{2n}}} = \lim_{n\to\infty}\frac{2n}{n^2} = 0$$

   The limit is less than 1, so the series converges (absolutely).

7. $\displaystyle\sum_{n=2}^\infty \frac{(-1)^{n-1}}{\sqrt{n}-1}$

   We see that $\sum 1/\sqrt{n}$ would diverge (and we should say so), so then we go to the Alternating Series Test.

   - $b_n = \frac{1}{\sqrt{n}-1}$, which is positive for $n \geq 2$.
   - b_n is decreasing, since the denominator is increasing.
   - $\lim_{n\to\infty} b_n = 0$

   By the Alternating Series Test, the series converges. Since it was not absolutely convergent, this means it is only conditionally convergent.