Final Exam Review
Calculus II
Sheet 2

1. True or False, and give a short reason:

   (a) The Ratio Test will not give a conclusive result for $\sum \frac{2n+3}{3n^4+2n^3+3n+5}$
   TRUE. The ratio test fails for $p-$like series (the limit will be 1). To show convergence, use a direct or limit comparison (Limit comparison with $1/n^3$)

   (b) If $\sum_{n=k}^{\infty} a_n$ converges for some large k, then so will $\sum_{n=1}^{\infty} a_n$.
   TRUE. The first few terms of a sum are irrelevant when looking at whether or not the sum converges (although they will effect what the sum converges to).

   (c) If f is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty f(x)\,dx$ converges.
   FALSE. For example, $1/(x-1)$. (The idea here is that functions must go to zero fast enough).

   (d) If f is continuous and $\displaystyle\int_0^9 f(x)\,dx = 4$, then $\displaystyle\int_0^3 xf(x^2)\,dx = 4$.
   FALSE.

$$\int_0^3 xf(x^2)\,dx \Rightarrow \begin{array}{rl} u & = x^2 \\ (1/2)\,du & = dx \\ x=0 & \Rightarrow u=0 \\ x=3 & \Rightarrow u=9 \end{array} \Rightarrow \frac{1}{2}\int_0^9 f(u)\,du = \frac{1}{2}\cdot 4 = 2$$

2. Short Answer:

   (a) Suppose the series $\sum c_n 3^n$ converges. Will $\sum c_n(-2)^n$ also converge? For what values of x will the series $\sum c_n(x-2)^n$ converge?
   SOLUTION: For the first part of the question, we can look as if it were a power series $\sum c_n x^n$ that converged at $x=3$. Therefore, the series would converge for all $|x| < 3$, and $x = -2$ is within that range. On the other hand, if we think of the series as $\sum c_n(x-2)^n$, then the series converges for all x so that $|x-2| < 3$, or at least within the interval $(-1, 5]$ (the convergence at $x-2 = 3$ might be conditional, that's why we did not include $x = -1$).

   (b) If $\sum a_n$, $\sum b_n$ are series with positive terms, and a_n, b_n both go to zero as $n \to \infty$, then what can we conclude if $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$?
   SOLUTION: We can conclude that the terms of $\sum a_n$ are going to zero faster than b_n. Thus, if $\sum b_n$ is convergent, so is $\sum a_n$, and if $\sum a_n$ is divergent, so is $\sum b_n$.

   (c) What is the derivative of $\sin^{-1}(x)$? Of $\tan^{-1}(x)$? What is the antiderivative of each?
   SOLUTION: The derivative of $\sin^{-1}(x)$ is $\frac{1}{\sqrt{1-x^2}}$. The derivative of $\tan^{-1}(x)$ is $\frac{1}{1+x^2}$
   To integrate either, use integration by parts. For $\sin^{-1}(x)$,

$$\begin{array}{ccc} + & \sin^{-1}(x) & 1 \\ - & 1/\sqrt{1-x^2} & x \end{array} \Rightarrow \int \sin^{-1}(x)\,dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

For this integral, use $u = 1 - x^2$, $du = -2x\,dx$ to get a final answer:

$$\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) + \sqrt{1-x^2} + C$$

(d) Find the sum: $\sum_{n=1}^{\infty} \mathrm{e}^{-2n}$
SOLUTION: The sum of a geometric series, in its general form is:

$$\sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r}$$

In this case, $r = \mathrm{e}^{-2}$, so the sum is: $\frac{\mathrm{e}^{-2}}{1+\mathrm{e}^{-2}}$

3. Suppose $h(1) = -2$, $h'(1) = 2$, $h''(1) = 3$, $h(2) = 6$, $h'(2) = 5$, and $h''(2) = 13$, and h'' is continuous. Evaluate $\int_1^2 h''(u)\,du$.

$$\int_1^2 h''(u)\,du = h'(2) - h'(1) = 5 - 2 = 3$$

4. Determine a definite integral representing: $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1+\frac{3i}{n}}$ [For extra practice, try writing the integral so that the right endpoint (or bottom bound) must be 5].
SOLUTION: We need to find f so that

$$f\left(5+\frac{3i}{n}\right) = \sqrt{1+\frac{3i}{n}}$$

Here is one: $f(x) = \sqrt{x-4}$. Our solution is:

$$\int_5^8 \sqrt{x-4}\,dx$$

5. Evaluate $\int_2^5 (1+2x)\,dx$ by using the definition of the integral (use right endpoints).
SOLUTION: The i^{th} right endpoint is $2+\frac{3i}{n}$. Evaluating f at this endpoint gives the following, from which we get the Riemann sum:

$$\left(1+2\left(2+\frac{3i}{n}\right)\right) = 1+4+\frac{6i}{n} = 5+\frac{6i}{n} \quad\Rightarrow\quad \sum_{i=1}^{n}\left(5+\frac{6i}{n}\right)\frac{3}{n}$$

Now break apart the sum to evaluate:

$$\lim_{n\to\infty}\frac{3}{n}\left(5\sum_{i=1}^{n}1+\frac{6}{n}\sum_{i=1}^{n}i\right) = \lim_{n\to\infty}\frac{3}{n}\left(5n+\frac{6}{n}\frac{n(n+1)}{2}\right) = \lim_{n\to\infty} 15+9\cdot\frac{n+1}{n} = 24$$

(Note that geometrically, the area of the trapezoid is also 24).

6. For each function, find the Taylor series for $f(x)$ centered at the given value of a:
SOLUTION:

(a) $f(x) = 1 + x + x^2$ at $a = 2$ We need $f(2), f'(2), f''(2)$: $f(2) = 7$. $f'(x) = 1 + 2x$, so $f'(2) = 5$. $f''(x) = 2$ Now,

$$1 + x + x^2 = 7 + 5(x-2) + \frac{2}{2!}(x-2)^2 = 7 + 5(x-2) + (x-2)^2$$

(b) $f(x) = \frac{1}{x}$ at $a = 1$. We need to compute derivatives:

n	$f^n(x)$	$f^n(1)$
0	x^{-1}	1
1	$-x^{-2}$	-1
2	$2x^{-3}$	2
3	$-(3\cdot 2)x^{-4}$	$-(3\cdot 2)$
4	$4\cdot 3\cdot 2x^{-5}$	$4\cdot 3\cdot 2$
$\vdots$	$\vdots$	$\vdots$
n	$(-1)^n n! x^{-(n+1)}$	$(-1)^n n!$

$$\Rightarrow \frac{f^{(n)}(1)}{n!} = (-1)^n \Rightarrow \sum_{n=0}^{\infty}(-1)^n(x-1)^n$$

Alternatively, we could use the geometric series:

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{n=0}^{\infty}(1-x)^n = \sum_{n=0}^{\infty}(-1)^n(x-1)^n$$

7. Find a so that half the area under the curve $y = \frac{1}{x^2}$ lies in the interval $[1, a]$ and half of the area lies in the interval $[a, 4]$.

SOLUTION: We could set this up multiple ways- here is one way to do it:

$$\int_1^a \frac{1}{x^2}\,dx = \frac{1}{2}\int_1^4 \frac{1}{x^2}\,dx \Rightarrow -\frac{1}{a} + 1 = \frac{3}{8} \Rightarrow a = \frac{8}{5}$$

8. Compute the limit, by using the series for $\sin(x)$: $\lim_{x\to 0} \frac{\sin(x)}{x}$

SOLUTION: The series for the sine function is:

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{(2n+1)}}{(2n+1)!} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

Therefore, the series for $\sin(x)/x$ is:

$$\frac{\sin(x)}{x} = 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 + \cdots$$

To find the limit as $x \to 0$, we can evaluate the series at $x = 0$, which leaves the limit as 1.

9. Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by $y = x$, $y = 4x - x^2$, about $x = 7$.

SOLUTION: First, find the region of interest. $y = 4x - x^2$ is an upside down parabola with $x-$intercepts at $x = 0, x = 4$. The point of intersection is $x = 4x - x^2 \Rightarrow 0 = 3x - x^2$, or $x = 0$ and $x = 3$. Now the region of interest is between $x = 0$, $x = 3$, above the line $y = x$ and below the parabola $y = 4x - x^2$. Rotate about $x = 7$, and we will use cylindrical shells (Washers would be possible, but messy!). The height of the cylinder is $(4x - x^2) - x = 3x - x^2$. The radius is $7 - x$. Therefore, the integral for the volume is:

$$\int_0^3 2\pi(7 - x)(3x - x^2)\, dx$$

10. Evaluate each of the following:

[The purpose of this problem is to get you to see the differences in notation]

(a) $\frac{d}{dx} \int_{3x}^{\sin(x)} t^3\, dt$. By FTC, part I: $\sin^3(x) \cdot \cos(x) - (3x)^3 \cdot 3$

(b) $\frac{d}{dx} \int_1^5 x^3\, dx = 0$ (this is the derivative of a constant)

(c) $\int_1^5 \frac{d}{dx} x^3\, dx = x^3 \Big|_1^5 = 5^3 - 1 = 124$. This is FTC, part II.

11. Converge (absolute or conditional) or Diverge?

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n+1)(n+2)}$ This will behave like $\sum(-1)^n \frac{1}{n}$, which only converges conditionally.

We can use the limit comparison test (with $\frac{1}{n}$) to show that the series does not converge absolutely:

$$\lim_{n\to\infty} \frac{n}{(n+1)(n+2)} \cdot \frac{n}{1} = 1$$

The two series will diverge together, so the given series diverges.

Now we use the Alternating Series Test to show that it converges conditionally: Each term is clearly positive, for $n > 0$. Is it decreasing?

$$f(x) = \frac{x}{(x+1)(x+2)} \qquad f'(x) = \frac{2 - x^2}{(x+1)^2(x+2)^2}$$

so the derivative is negative for $x > \sqrt{2}$ (or the terms of the series are decreasing for $n > 2$). Finally, show that the terms are going to zero:

$$\lim_{n\to} \frac{n}{(n+1)(n+2)} = \lim_{n\to\infty} \frac{n}{n^2 + 3n + 2} = \lim_{n\to\infty} \frac{1}{2n+3} = 0$$

(the last equality by l'Hospital's rule).

(b) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2 - 1}}{n^3 + 2n^2 + 5}$

It looks like it should converge by comparing it to $\sum \frac{1}{n^2}$, so we'll try the limit comparison test:

$$\lim_{n\to\infty} \frac{\sqrt{n^2 - 1}}{n^3 + 2n^2 + 5} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{\sqrt{n^2 - 1}}{n^3 + 2n^2 + 5} \cdot \frac{\sqrt{n^4}}{1} = \lim_{n\to\infty} \frac{\sqrt{n^6 - n^4}}{n^3 + 2n^2 + 5}$$

(Don't use l'Hospital's rule!) Divide top and bottom by n^3:

$$\lim_{n\to\infty} \frac{\sqrt{1-\frac{1}{n^2}}}{1+\frac{2}{n}+\frac{5}{n^3}} = 1$$

By the limit comparison test, the given series converges (absolutely, but that is irrelevant since the terms are all positive anyway).

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{4^k+k}{k!}$ Use the ratio test:

$$\frac{4^{k+1}+(k+1)}{(k+1)!} \cdot \frac{k!}{4^k+k} = \frac{4^{k+1}+k+1}{(k+1)(4^k+k)} = \frac{4+\frac{k}{4^k}+\frac{1}{4^k}}{(k+1)(1+\frac{k}{4^k})}$$

The numerator approaches 4 as $k \to \infty$ and the denominator goes to ∞ as $k \to \infty$, so overall, the limit is 0. Therefore, this series converges (absolutely) by the Ratio Test.

12. Find the interval of convergence.

(a) $\displaystyle\sum_{n=1}^{\infty} n^n x^n$ By the root test, $\displaystyle\lim_{n\to\infty} (n^n x^n)^{1/n} = \lim_{n\to\infty} nx = \infty$ Therefore, the only point of convergence is when $x = 0$. (The radius of convergence is also 0).

Note: The root test is not used very often, but in this situation (where everything is raised to the n^{th} power), this will make quick work of the problem.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{n4^n}$

Use the Ratio Test, as usual:

$\displaystyle\lim_{n\to\infty} \frac{|x+2|^{n+1}}{(n+1)4^{n+1}} \cdot \frac{n4^n}{|x|^n} = \lim_{n\to\infty} \frac{n+1}{n}\frac{|x+2|}{4} = \frac{|x+2|}{4} < 1$ This means that the radius of convergence is 4, and the interval so far is $(-6, 2)$.

Check the endpoints: If $x = 2$, then the sum is $\sum \frac{1}{n}$ which diverges. If $x = -6$, then the sum is $\sum \frac{(-1)^n}{n}$, which converges. The interval of convergence is therefor $-6 \leq x < 2$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}}$

Use the Ratio Test:

$\displaystyle\lim_{n\to\infty} \frac{2^{n+1}|x-3|^{n+1}}{\sqrt{n+4}} \cdot \frac{\sqrt{n+3}}{2^n|x-3|^n} = \lim_{n\to\infty} \sqrt{\frac{n+3}{n+4}} \cdot 2|x-3| = 2|x-3| < 1$

Therefore, the radius of convergence is $1/2$ and the interval is $5/2 < x < 7/2$. Now check endpoints:

If $x = \frac{5}{2}$, the sum becomes $\sum \frac{(-1)^n}{\sqrt{n+3}}$, which converges by the Alternating Series test, and if $x = \frac{7}{2}$, the sum becomes $\sum \frac{1}{\sqrt{n+3}}$ which diverges (p-series).

13. Evaluate:

(a) $\int_0^\infty \frac{1}{(x+2)(x+3)}\,dx$ By partial fractions,

$$\int \frac{1}{(x+2)(x+3)}\,dx = \int \frac{1}{x+2} - \frac{1}{x+3}\,dx = \ln|x+2| - \ln|x+3| = \ln\left|\frac{x+2}{x+3}\right|$$

As $x \to \infty$, $\ln\left|\frac{x+2}{x+3}\right| \to \ln(1) = 0$. Altogether we get:

$$\int_0^\infty \frac{1}{(x+2)(x+3)}\,dx = 0 - \ln(2/3) = \ln(3/2)$$

(b) $\int u(\sqrt{u} + \sqrt[3]{u})\,du$ Simplify algebraically first, to get $\int u^{3/2} + u^{4/3}\,du = \frac{2}{5}u^{5/2} + \frac{3}{7}u^{7/3} + C$

(c) $\int \frac{x^2}{(4-x^2)^{3/2}}\,dx$
Use a triangle whose hypotenuse is 2, side opposite θ is x, and side adjacent is $\sqrt{4-x^2}$. Then, substitute $2\sin(\theta) = x$, $2\cos(\theta) = \sqrt{4-x^2}$, and we get:

$$\int \frac{4\sin^2(\theta)\cdot 2\cos(\theta)}{2^3\cos^3(\theta)}\,d\theta = \int \tan^2(\theta)\,d\theta = \int \sec^2(\theta) - 1\,d\theta = \tan(\theta) - \theta$$

Convert back using triangles to get: $\frac{x}{\sqrt{4-x^2}} - \sin^{-1}(x/2) + C$

(d) $\int \frac{\tan^{-1}(x)}{1+x^2}\,dx$ Let $u = \tan^{-1}(x)$, so $du = \frac{1}{1+x^2}\,dx$. Then the integral becomes

$$\int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\tan^{-1}(x))^2 + C$$

(e) $\int \frac{1}{\sqrt{x^2-4x}}\,dx$
"Complete the Square" in the denominator to get $x^2 - 4x = (x-2)^2 - 4$. Now, use a triangle whose hypotenuse is $x-2$, side adjacent is 2, and side opposite is $\sqrt{(x-2)^2 - 2^2}$. Then,

$$2\tan(\theta) = \sqrt{(x-2)^2 - 2^2}, \quad 2\sec(\theta) = x - 2, \quad 2\sec(\theta)\tan(\theta)d\theta = dx$$

Substituting, we get:

$$\int \frac{1}{\sqrt{x^2-4x}}\,dx = \int \frac{2\sec(\theta)\tan(\theta)}{2\tan(\theta)}\,d\theta = \int \sec(\theta)\,d\theta = \ln|\sec(\theta) + \tan(\theta)| + C$$

[NOTE: You'll be given the formulas as on the previous exam]. Final answer:

$$\ln\left|\frac{x-2}{2} + \frac{\sqrt{(x-2)^2-4}}{2}\right| + C$$

(f) $\int x^4 \ln(x)\, dx$ Use integration by parts

$$\begin{array}{ccc} + & \ln(x) & x^4 \\ - & 1/x & (1/5)x^5 \end{array} \Rightarrow \frac{1}{5}x^5\ln(x) - \frac{1}{5}\int x^4\,dx = \frac{1}{5}x^5\ln(x) - \frac{1}{25}x^5 + C$$

(g) $\int \mathrm{e}^{-x}\sin(2x)\,dx$. This is the type of integral for which we perform integration by parts twice to get the same integral on both sides of the equation:

$$\left|\begin{array}{c|c|c} + & \sin(2x) & \mathrm{e}^{-x} \\ - & 2\cos(2x) & -\mathrm{e}^{-x} \\ + & -4\sin(2x) & \mathrm{e}^{-x} \end{array}\right| \Rightarrow \int \mathrm{e}^{-x}\sin(2x)\,dx = -\mathrm{e}^{-x}\sin(2x) - 2\mathrm{e}^{-x}\cos(2x) - 4\int \mathrm{e}^{-x}\sin(2x)\,dx$$

so that

$$\int \mathrm{e}^{-x}\sin(2x)\,dx = -\frac{1}{5}\mathrm{e}^{-x}\sin(2x) - \frac{2}{5}\mathrm{e}^{-x}\cos(2x)$$

(h) $\int_0^3 \frac{1}{\sqrt{x}}\,dx$

Note that we have a vertical asymptote at $x = 0$, so

$$\int_0^3 \frac{1}{\sqrt{x}}\,dx = \lim_{T\to 0^+}\int_T^3 x^{-1/2}\,dx = \lim_{T\to 0^+} 2x^{1/2}\Big|_T^3 = 2\sqrt{3} - 0 = 2\sqrt{3}$$

(i) $\int \sin^2 x\cos^5 x\ dx$ Recall our rules for dealing with powers of sine and cosine: If both are even, use the formulas for $\sin^2(x)$ and $\cos^2(x)$. If one (or both) are odd, try substitution:

$$\int \sin^2(x)\cos^4(x)\cdot\cos(x)\,dx$$

which means we want to write $u = \sin(x)$. Use the Pythagorean Identity: $\cos^4(x) = (1 - \sin^2(x))^2$, so that:

$$\int \sin^2(x)\cos^4(x)\cdot\cos(x)\,dx = \int \sin^2(x)\left(1 - \sin^2(x)\right)^2\cdot\cos(x)\,dx = \int u^2(1-u^2)^2\,du$$

Simplify this last integral, and integrate:

$$\int u^6 - 2u^4 + u^2\,du = \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C$$

so our final answer is:

$$\frac{1}{7}\sin^7(x) - \frac{2}{5}\sin^5(x) + \frac{1}{3}\sin^3(x) + C$$

14. A leaky 10-kg bucket is lifted from the ground to a height of 12 meters at a constant speed with a rope has density 0.8 kg/m. Initially the bucket contains 36 kg of water, but the water leaks at a constant rate and finishes draining just as the bucket reaches the 12 meter level. Set up the integral to compute how much work is done (gravity is 9.8 m/s^2):

SOLUTION: At a height of x meters:

- The mass of the rope is $0.8(12 - x)$ kg.
- The mass of the water is $(36/12)(12 - x) = (36 - 3x)$ kg.
- The mass of the bucket is constant at 10 kg.

The work needed to lift the rope a small unit, dx, is the mass times g, or:

$$9.8 \int_0^{12} 0.8(12 - x) + (36 - 3x) + 10\,dx$$

Integrate this out to get approximately 3857 J.

15. Prove the following by induction:

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$$

SOLUTION:

- Prove it for a first case: If $n = 1$, the

$$1 \cdot 2 = \frac{1 \cdot 2 \cdot 3}{3}$$

  Which is true.

- Assume the statement is true for $n = k$, then use that to prove it true for $n = k+1$: Assume true for $n = k$:

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k \cdot (k+1) = \frac{k(k+1)(k+2)}{3}$$

  And, we want to show that this implies that:

$$1 \cdot 2 + 2 \cdot 3 + \cdots + k \cdot (k+1) + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

  So, starting with the left side of the equation, we want to get the right side. As is our usual practice, break up the sum to use the assumption:

$$1 \cdot 2 + 2 \cdot 3 + \cdots + k \cdot (k+1) + (k+1)(k+2) =$$

$$[1 \cdot 2 + 2 \cdot 3 + \cdots + k \cdot (k+1)] + (k+1)(k+2) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

  Factor out $(k+1)(k+2)$

$$= (k+1)(k+2)\left(\frac{k}{3} + 1\right) = \frac{(k+1)(k+2)(k+3)}{3}$$

  Therefore, the statement is true for all positive integers n.