Selected Solutions, Section 5.2

1. This is good practice in taking *left* endpoints.

   In this case, $f(x) = 3 - x/2$, and the interval is $[2, 14]$. The Riemann sum using 6 rectangles will use:

   - Width of each rectangle: $(14 - 2)/6 = 12/6 = 2$.
   - The height of the rectangles will be evaluated at left endpoints. Subdividing the interval using 6 equal subintervals gives us endpoints:

     $$2, 4, 6, 8, 10, 12, 14$$

     so that "left endpoints" are $2, 4, 6, 8, 10, 12$.
   - The area is computed as $f(x_i^*) = 3 - x_i^*/2$, or using left endpoints:

     $$(2 + 1 + 0 - 1 - 2 - 3) \cdot 2 = -6$$

8. The table is used to estimate $\int_3^9 f(x)\,dx$ using 3 equal subintervals. The intervals then have endpoints

   $$3, 5, 7, 9$$

   (a) Using right endpoints, the area is approximately:

   $$((-0.6) + (0.9) + (1.8))2 = 4.2$$

   Since f is increasing, right endpoints give an overestimate.

   (b) Using left endpoints, the area is approximately:

   $$((-3.4) + (-0.6) + (0.9))2 = -6.2$$

   Left endpoints give an underestimate for an increasing function.

   (c) Using midpoints:

   $$((-2.1) + (0.3) + (1.4))2 = -0.8$$

   This is probably more accurate than the other two.

18. For problems 17-19, we want to compare what is given to the formula in the text,

    $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

    In this particular case, the expression looks like

    $$\int_{\pi}^{2\pi} \frac{\cos(x)}{x}\,dx$$

23. This is similar to the one we did in class (we did $\int x^2+1\,dx$). "Theorem 4" uses right endpoints.

Before getting started, note that the widths are $(0--2)/n = 2/n$, and the "i^{th} right endpoint" is $-2+2i/n$. Therefore,

$$\int_{-2}^{15}(x^2+x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\left(-2+\frac{2i}{n}\right)^2+\left(-2+\frac{2i}{n}\right)\right)\cdot\frac{2}{n}$$

Simplify those heights first by expanding the product out and simplifying. One way is to evaluate x^2+x as $x(x+1)$:

$$\left(-2+\frac{2i}{n}\right)\left[\left(-2+\frac{2i}{n}\right)+1\right] = \left(-2+\frac{2i}{n}\right)\left(-1+\frac{2i}{n}\right) = 2-\frac{6i}{n}+\frac{4i^2}{n^2}$$

Multiply by $2/n$ and bring in the sum:

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{4}{n}-\frac{12i}{n^2}+\frac{8i^2}{n^3}\right) = \lim_{n\to\infty}\left[\frac{4}{n}\sum_{i=1}^{n}1-\frac{12}{n^2}\sum_{i=1}^{n}i+\frac{8}{n^3}\sum_{i=1}^{n}i^2\right]$$

$$= \lim_{n\to\infty}\left[4-\frac{12}{n^2}\cdot\frac{n(n+1)}{2}+\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right] = 4-6+\frac{8}{3} = \frac{2}{3}$$

27. This exercise is very similar to the last one. Using right endpoints, the width of each rectangle is $(b-a)/n$. The i^{th} right endpoint is $a+i(b-a)/n$, and in this case, $f(x)=x$. Therefore,

$$\int_a^b x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(a+\frac{(b-a)i}{n}\right)\left(\frac{b-a}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{a(b-a)}{n}+\frac{(b-a)^2 i}{n^2}\right)$$

$$= \lim_{n\to\infty}\left[\frac{a(b-a)}{n}\sum_{i=1}^{n}1+\frac{(b-a)^2}{n^2}\sum_{i=1}^{n}i\right] = \lim_{n\to\infty}\left[a(b-a)+\frac{(b-a)^2}{n^2}\cdot\frac{n(n+1)}{2}\right]$$

$$= a(b-a)+\frac{(b-a)^2}{2} = (b-a)\left(a+\frac{b-a}{2}\right) = \frac{1}{2}(b-a)(b+a) = \frac{b^2-a^2}{2}$$

37. For #37, recall that $x^2+y^2=r^2$ is a circle of radius r centered at the origin. Solving for y, we could express the upper half of a circle of radius r as:

$$y=\sqrt{r^2-x^2}$$

Therefore, the function $f(x)=1+\sqrt{9-x^2}$ is the upper half of a circle of radius 3 (centered at the origin) and shifted up one unit. The integral is using the interval $[-3,0]$, so the area in this case,

$$\int_{-3}^{0}(1+\sqrt{9-x^2})\,dx$$

is the area of 1/4 of the circle, plus the rectangle of height 1:

$$\int_{-3}^{0}(1+\sqrt{9-x^2})\,dx = 1\cdot 3+\frac{1}{4}\pi 3^2 = 3+\frac{9}{4}\pi$$

53. We want to re-write the integral as

$$\int_{-4}^{3} f(x)\,dx + 2\int_{-4}^{3} x\,dx + \int_{-4}^{3} 5\,dx$$

and use geometry for the last two integrals. The first/last are easy to compute. For the middle one, draw a sketch and use the area of a triangle. You should find that (in order) the values are:

$$-3 - 12 + 30 = 15$$

55. For this exercise, note that $f(x) = x^2 - 4x + 4 = (x-4)^2$, so that $f(x) \geq 0$. Therefore,

$$\int_{0}^{4} f(x)\,dx \geq 0 \cdot (4-0) = 0$$

59. To use Property 8, we have to find the minimum m and maximum M of $\sqrt{x}$ on the interval $[1,4]$. Since the function is increasing, the minimum is $\sqrt{1} = 1 = m$ and the maximum is $\sqrt{4} = 2 = M$. Therefore,

$$1(4-1) \leq \int_{1}^{4} \sqrt{x}\,dx \leq 2(4-1)$$

or

$$3 \leq \int_{1}^{4} \sqrt{x}\,dx \leq 6$$