Selected Solutions, Section 5.3

4. This is a good exercise to understand the "area function" that we described in class.

   (a) $g(0) = \int_0^0 f(t)\,dt = 0$, and $g(6) = \int_0^6 f(t)\,dt = 0$ by symmetry (it looks like there is as much positive "area" as negative.

   (b) Estimate $g(x)$ for $x = 1, 2, 3, 4, 5$.
   As estimates, we might have something like (respectively):
   $$2.8, 4.9, 5.7, 4.9, 2.8$$

   (c) As we go from $x = 0$ to $x = 3$, we are adding area, so g is increasing.

   (d) g increases on $(0, 3)$ and decreases on $(3, 6)$, so g has a (global) maximum at $x = 3$.

   (e) The graph of g must have a max at $x = 3$, symmetric about $x = 3$, and begin and end on the $x-$axis.

5. For the sketch, the "area function" should be zero at $x = 1$, since
$$g(1) = \int_1^1 t^2\,dt = 0$$
Furthermore, $g(0) = \int_1^0 t^2\,dt = -\int_0^1 t^2\,dt$, so $g(0) < 0$. In fact, you should find that the area function is given by
$$g(x) = \frac{1}{3}x^3 - \frac{1}{3}$$
so that $g'(x) = x^2$.

HINT for 7-18: First, define
$$g(x) = \int_a^x f(t)\,dt$$
Then, if you have a function of x (call it $h(x)$) as a limit of integration, the integral is actually a *composition:*
$$\int_a^{h(x)} f(t)\,dt = g(h(x))$$
Therefore, the derivative is:
$$\frac{d}{dx}\int_a^{h(x)} f(t)\,dt = g'(h(x))h'(x) = f(h(x))h'(x)$$

17. Using the hint, we define
$$g(x) = \int_1^x \frac{u^3}{1+u^2}\,du$$
Then
$$\int_{1-3x}^1 \frac{u^3}{1+u^2}\,du = -\int_1^{1-3x} \frac{u^3}{1+u^2}\,du = -g(1-3x)$$
Differentiate both sides, and we get:
$$\frac{d}{dx}\int_{1-3x}^1 \frac{u^3}{1+u^2}\,du = -g'(1-3x)(-3) = 3g'(1-3x) = 3\frac{(1-3x)^3}{1+(1-3x)^2}$$

27. Hint: Multiply the integrand out before antidifferentiating.

29. Hint: Do some algebra first-

$$\frac{x-1}{\sqrt{x}} = x^{-1/2}(x-1) = x^{1/2} - x^{-1/2}$$

33. Hint: Multiply out the integrand before antidifferentiating.

35. Hint: Simplify like #29 above before antidifferentiating.

39. Hint: What is the derivative of $8\tan^{-1}(x)$?

41. Hint: $\mathrm{e}^{u+1} = \mathrm{e}^u \mathrm{e}^1$

57. Given that

$$F(x) = \int_x^{x^2} \mathrm{e}^{t^2}\,dt$$

find $F'(x)$.

SOLUTION: Do something like we hinted at before- First define

$$g(x) = \int_a^x \mathrm{e}^{t^2}\,dt$$

where a is any real number. Then

$$F(x) = \int_x^{x^2} \mathrm{e}^{t^2}\,dt = \int_x^a \mathrm{e}^{t^2}\,dt + \int_a^{x^2} \mathrm{e}^{t^2}\,dt = -\int_a^x \mathrm{e}^{t^2}\,dt + \int_a^{x^2} \mathrm{e}^{t^2}\,dt = -g(x) + g(x^2)$$

so that

$$F'(x) = -g'(x) + g'(x^2)(2x) = -\mathrm{e}^{x^2} + 2x\mathrm{e}^{x^2}$$

61. The idea here is that, if you have

$$g(x) = \int_a^x f(t)\,dt$$

then $g'(x) = f(x)$ and $g''(x) = f'(x)$. Therefore, g will be concave down when $g'' < 0$. In this particular case, we have to use the quotient rule and differentiate:

$$y'' = \frac{d}{dx}\left(\frac{x^2}{x^2+x+2}\right) = \frac{(2x)(x^2+x+2) - x^2(2x+1)}{(x^2+x+2)^2} = \cdots = \frac{x(x+4)}{(x^2+x+2)^2}$$

Therefore, $y'' < 0$ where $x(x+4) < 0$. Give that a quick sketch- It's an upside down parabola, and we see that $y'' < 0$) when x is in the interval $(-4, 0)$.

63. If $f(1) = 12$, f' is continuous and $\int_1^4 f'(x)\,dx = 17$. What is $f(4)$.

SOLUTION: First, recognize that $f(x)$ is an antiderivative of $f'(x)$. Then, by the FTC (Fundamental Theorem of Calculus), we know that

$$\int_1^4 f'(x)\,dx = f(4) - f(1) \quad \Rightarrow \quad 17 = f(4) - 12 \quad \Rightarrow \quad f(4) = 29$$

67. We want to use the fact that, if $h'(x) < 0$, then $h(x)$ is decreasing, and if $h'(x) > 0$, then $h(x)$ is increasing. Also, if $h'(a) = 0$ and $h'(x)$ goes from negative to positive close to $x = a$, then $h(a)$ is a local minimum. Similarly, for a local max, $h'(a) = 0$ and $h'(x)$ goes from positive to negative.

    You now want to think of the graph as the graph of the derivative of some function. That is, $f(t) = h'(t)$, and the antiderivative is $\int_0^x f(t)\,dt = h(x)$ where $h(0) = 0$.

    Now, at $t = 0, 3, 7$, the graph of the derivative (f) is going from negative to positive, so at these points, the original function has local minima- We should also exclude $t = 0$, since we do not know what happens if $t < 0$.

    At times $t = 1, 5, 9$, the graph is going from positive to negative, so when the derivative does that, the original function has local maxima; although again we should exclude $t = 9$ from that list.

    The most positive value of the function would be at $t = 9$, so that would be the "global" or "absolute" maximum.

    Finally, g is concave down where $f'(t) < 0$ (or where the graph of f is decreasing). These intervals would be $(1/2, 2)$, $(4, 6)$, $(8, 9)$.

69. Since we're told the interval is $[0, 1]$, re-write the expression to really look like a Riemann sum. We want the i^{th} right endpoint to be i/n, and the width of each rectangle to be $1/n$:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^3} \cdot \frac{1}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \cdot \frac{1}{n} = \int_0^1 x^3\,dx = \frac{1}{4}$$