11.8 Selected Soln's:

#10 $\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$ Find the radius & interval of convergence.

Use the Ratio Test; Simplify before taking the limit:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{10^{n+1}|x|^{n+1}}{(n+1)^3} \cdot \frac{n^3}{10^n|x|^n} = \left(\frac{n}{n+1}\right)^3 \cdot 10 \cdot |x|$$

So $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^3 \cdot 10 \cdot |x| = 10|x|.$

To converge, the limit is less than 1: $10|x| < 1 \Rightarrow |x| < 1/10$

The radius of convergence $\rho = \frac{1}{10}$.

To find the interval, check the endpts: $x = -\frac{1}{10}, \frac{1}{10}$:

At $x = \pm 1/10$:

$$\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3} = \sum_{n=1}^{\infty} \frac{10^n \left(\frac{\pm 1}{10}\right)^n}{n^3} = \sum_1^{\infty} \frac{(\pm 1)^n}{n^3}$$

conv. p-series. for both.

The interval: $[-1/10, \frac{1}{10}]$.

11.8

\# 22 $\displaystyle\sum_{n=2}^{\infty} \frac{b^n}{\ln(n)} (x-a)^n, \quad b>0$

Ratio Test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{b^{n+1}|x-a|^{n+1}}{\ln(n+1)} \cdot \frac{\ln(n)}{b^n|x-a|^n}$$

$$= \frac{\ln(n)}{\ln(n+1)} \cdot b \cdot |x-a|$$

Take the limit as $n \to \infty$: $\displaystyle\lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)} = 1$, so

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = b \cdot |x-a| < 1 \Rightarrow |x-a| < \frac{1}{b}$$

$\boxed{\text{The radius is } \frac{1}{b}.}$

For the interval of conv, check $x = a - 1/b$ and $x = a + 1/b$

At $x = a - 1/b$, $\displaystyle\sum \frac{b^n(x-a)^n}{\ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$

~~Since~~ The series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges (compare w/ $\frac{1}{n}$)

but we can use Alt Series Test for $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$,

which converges (conditionally).

The series converges at $x = a - 1/b$, diverges at $x = a + 1/b$

$\Rightarrow$ $\boxed{[a - 1/b,\ a + 1/b) \text{ is the interval of conv}}$

#24 $\quad \sum \frac{n^2 x^n}{2\cdot 4\cdot 6\cdots(2n)}$

Ratio Test:

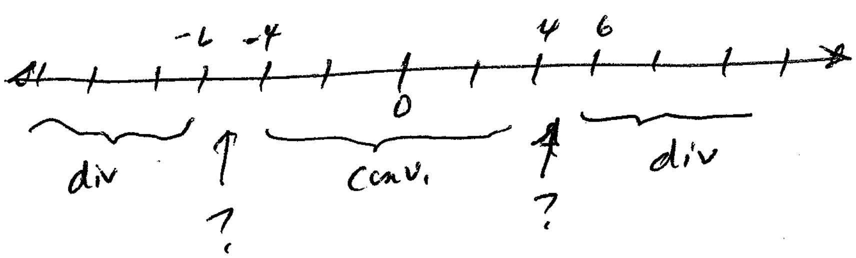
$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2\,|x|^{n+1}}{2\cdot 4\cdot 6\cdots(2n)(2n+2)} \cdot \frac{2\cdot 4\cdot 6\cdots(2n)}{n^2\,|x|^n}$$

$$= \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{2n+2}\,|x|$$

And the limit of this is 0 for all x. (So the radius of conv is $\rho = \infty$).

#30 Use a number line to help you - We know that the interval of convergence is a symmetric interval about the center point (not counting endpts).

From what is given, $\sum c_n x^n$ converges when $x=-4$ and diverges when $x=6$, we can conclude the following picture:



Therefore, (a) has $x=1$ (conv)

(b) has $x=8$ (div)

(c) $x=-3 \Rightarrow$ conv

(d) $x=-9 \Rightarrow$ div

#31 Radius of conv for $\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$ (k is a fixed #)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{[(n+1)!]^k \, |x|^{n+1}}{(k(n+1))!} \cdot \frac{(kn)!}{(n!)^k \, |x|^n}$$

$$= \left(\frac{(n+1)!}{n!}\right)^k \cdot \frac{(kn)!}{(kn+k)!} \cdot |x|$$

$$= (n+1)^k \cdot \frac{1}{(kn+1)(kn+2)\cdots(kn+k)} \cdot |x|$$

Now, consider the fraction:

$$\frac{(n+1)^k}{(kn+1)(kn+2)\cdots(kn+k)}$$

The dominating term in the numerator (if expanded) would be n^k. In the denominator, we are mult. kn times itself k times.

Therefore,

$$\frac{(n+1)^k}{(kn+1)\cdots(kn+k)} \approx \frac{n^k}{(kn)^k} \xrightarrow{n\to\infty} \frac{1}{k^k}$$

The series converges if $\frac{1}{k^k}|x| < 1 \Rightarrow |x| < k^k$

33. No, since the interval must be symmetric about some point. The given interval is half-infinite.

34. You can see these on Wolfram Alpha -

plot $1/(1-x)$ and $1+x+x^2+x^3$ (then keep adding more terms)