Proof by Induction

This document is meant to provide you with a short introduction to "proof by induction", and will give you a chance to practice the technique with some exercises.

What is it?

We have some statement (or formula) that depends on n. We would like to prove that the statement is true for all natural numbers $n = 1, 2, 3, \cdots$.

The Induction Principle: Let $P(n)$ be a statement which depends on $n = 1, 2, 3, \cdots$. Then $P(n)$ is true for all n if:

- $P(1)$ is true (the *base case*.
- Prove that $P(k)$ is true implies that $P(k+1)$ is true. This is sometimes broken into two steps, but they go together: Assume that $P(k)$ is true, then show that with this assumption, $P(k+1)$ must be true.

What is the logic here? Think of the statements $P(1), P(2), P(3)$, etc. as dominoes. If you're able to drop (prove true) the first domino, and dropping one means the one next to it must drop, then all statements drop (are true). An example may clear things up:

Example:

Prove that the sum of the first n numbers, is the given formula:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

This is our statement, $P(n)$. Now follow the steps we outlined:

- Prove that $P(1)$ is true: $1 = \frac{1(2)}{2}$ is true.
- Now we assume that the statement is true using $n = k$. So assume $P(k)$ is true:

  $$1 + 2 + 3 \cdots + k = \frac{k(k+1)}{2}$$

  And prove this implies that $P(k+1)$ must be true:

  $$1 + 2 + 3 \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

  Proof: Begin with the sum, and keep in mind the formula we want:

  $$1 + 2 + \cdots + k + (k+1)$$

  By our assumption, this can be written as:

  $$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

Simplify the right hand side:

$$\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+2)(k+1)}{2}$$

which is what we wanted. Conclusion: The formula

$$\sum_{i=1}^{n} = \frac{n(n+1)}{2}$$

is true for all natural numbers n.

One more worked example:

Prove by induction that

$$1 + 3 + 5 + 7 + \cdots + (2n-1) = n^2$$

To prove this by induction, we work through the steps:

- Is $P(1)$ true? $1 = 1$ is kind of trivial. You can do another one if you like, with $n = 2$: $1 + 3 = 2^2$....
- Assume $P(k)$ is true, then show $P(k+1)$ is true: Assume

  $$\sum_{i=1}^{k} (2i-1) = k^2$$

  Show that this implies

  $$\sum_{i=1}^{k+1} (2i-1) = (k+1)^2$$

  Proof: From what we assumed, we can write:

  $$\sum_{i=1}^{k+1} (2i-1) = \sum_{i=1}^{k} (2i-1) + (2k+1) = k^2 + 2k + 1 = (k+1)^2$$

  And we're done. Conclusion: The summation formula

  $$\sum_{i=1}^{n} (2i-1) = n^2$$

  is true for all $n = 1, 2, 3, \cdots$

Exercises

Prove the following are true by induction:

1. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
2. $\displaystyle\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$
3. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$