Solutions to the Review Questions, Exam 3

1. A cross section of a tank of water is the bottom half of a circle of radius 10 ft, and is 50 ft long. Find the work done in pumping the water over the rim of the tank if it filled to a depth of 7 feet (set up the integral only, water weighs 62.5 lbs per cubic feet.) Set up the integral if we were pumping the water up an additional 10 feet up.

   SOLUTION: The actual height of the water using a standard coordinate system would be $h = 10 - y$, so that the "width" is $2\sqrt{100 - y^2}$ and length is 50. The distance pumped would then be $-y$:

   $$W = 62.5 \int_{-10}^{-3} 2 \cdot 50 \cdot \sqrt{100 - y^2}(-y)\, dy$$

   (This may change depending on how the original circle was laid out).

2. A heavy rope, 20 meters long, weighs 0.5 kg/m and hangs over a building that is 40 meters tall.

   (a) How much work is done pulling the rope to the top?

   SOLUTION: Let x be the meters to the top of the building. Then a small portion of rope dx units weighs $9.8 \cdot \frac{1}{2}\, dx$ newtons, and it travels x meters. Therefore, the overall work is

   $$\int_0^{20} 4.9x\, dx = 980 \text{ newton-meters}$$

   (b) How much work is done pulling half of the rope to the top? (Hint: It makes sense that it is not half your previous answer, right?)

   SOLUTION: We haul up the top half of the rope, with the bottom half of the rope coming along for the ride. The work for the top half:

   $$\int_0^{10} 4.9x\, dx = 245 \text{ newton-meters}$$

   And the bottom half can be thought of as a single thing- the bottom half is 10 meters, with $g = 9.8$ and density of $1/2$, it weighs 5 kg $\times 9.8$, and it all traveled 10 meters. So the work hauling up the bottom half of the rope is 490.

   Altogether, it took 735 N-m to haul up half the rope (the second half would be easier!).

3. Write the partial fraction decomposition for each of the following (do not actually solve for the coefficients):

   (a) $\dfrac{3 - 4x^2}{(2x+1)^3} = \dfrac{A}{2x+1} + \dfrac{B}{(2x+1)^2} + \dfrac{C}{(2x+1)^3}$

   (b) $\dfrac{7x - 41}{(x-1)^2(2-x)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{2-x}$

   (c) $\dfrac{x+1}{x^3(x^2 - x + 10)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{Dx + E}{x^2 - x + 10} + \dfrac{Fx + G}{(x^2 - x + 10)^2}$

   We note that $x^2 - x + 10$ is irreducible, since $b^2 - 4ac = 1 - 4 \cdot 10 < 0$.

4. Integrate the following:

$$\int \frac{2x^3 - x^2 - 4x - 13}{x^2 - x - 2}\, dx$$

SOLUTION: Do long division first:

$$\frac{2x^3 - x^2 - 4x - 13}{x^2 - x - 2} = 2x + 1 + \frac{x - 11}{x^2 - x - 2} = 2x + 1 + \frac{x - 11}{(x+1)(x-2)}$$

Now expand the last term:

$$\frac{x - 11}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

Solve for A, B: $x - 11 = A(x-2) + B(x+1)$. If we substitute $x = -1$, we get $-12 = -3A$, or $A = 4$. If we substitute $x = 2$, we get $-9 = 3B$, or $B = -3$. Therefore,

$$\frac{2x^3 - x^2 - 4x - 13}{x^2 - x - 2} = 2x + 1 + 4\,\frac{1}{x+1} - 3\,\frac{1}{x-2}$$

and the integral is

$$x^2 + x + 4\ln|x+1| - 3\ln|x-2| + C$$

5. If $x = \tan(\theta)$, show that $\sin(2\theta) = \frac{2x}{1+x^2}$.

We run into something similar to this when we integrate using a trig substitution. In this case, use a reference triangle for the tangent, and note that

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2 \cdot \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}} = \frac{2x}{1+x^2}$$

6. Find the length of the arc of the curve $y = x^{3/2}$ from the point $(1,1)$ to $(4,8)$.

SOLUTION: Note that the given $y-$values are not necessary; we only need $1 \le x \le 4$. Compute the integrand for the arc length formula:

$$\sqrt{1 + (y')^2} = \sqrt{1 + ((3/2)x^{1/2})^2} = \sqrt{1 + \frac{9}{4}x} = \frac{1}{2}\sqrt{4 + 9x}$$

Now integrate from 1 to 4:

$$\frac{1}{2}\int_1^4 \sqrt{4+9x}\, dx = \frac{1}{2} \cdot \frac{1}{9}\int_{14}^{40} u^{1/2}\, du = \frac{1}{27}\left(40^{3/2} - 14^{3/2}\right)$$

7. Show that $\int x f''(x)\, dx = x f'(x) - f(x)$

This is integration by parts:

$$\begin{array}{ccc} + & x & f''(x) \\ - & 1 & f'(x) \\ + & 0 & f(x) \end{array} \quad \Rightarrow \quad x f'(x) - f(x)$$

8. True or False? (And give a short reason)

   (a) To find $\int \sin^2(x)\cos^5(x)\,dx$, rewrite the integrand as $\sin^2(x)(1-\sin^2(x))^2\cos(x)$
   SOLUTION: That is true; then let $u=\sin(x)$ and $du=\cos(x)\,dx$.

   (b) Integration by parts is the integral version of the Product Rule for derivatives.
   SOLUTION: That is true. We showed it in class, but you could also start with the product rule, then integrate both sides:

$$(fg)' = f'g + fg' \quad \rightarrow \quad f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

   (c) To find $\int \frac{2x-3}{x^2-3x+5}\,dx$, start by completing the square in the denominator.
   SOLUTION: False- Start by looking for an obvious u, du substitution- In this case, $u = x^2-3x+5$ and $du = 2x-3\,dx$

   Side Note: Would it be wrong to complete the square? You would get to the same answer, but it would take you significantly more time...

   (d) To find $\int \frac{3}{x^2-3x+5}\,dx$, start by completing the square in the denominator.
   SOLUTION: False. Start by checking that you cannot factor the denominator- In this case, we cannot, so then continue by completing the square.

   (e) To find $\int \frac{3}{x^2-4x+3}\,dx$, start by completing the square in the denominator.
   SOLUTION: False. Start by checking the denominator- In this case, we can factor it, so we should do that and use partial fractions.

   (f) u, du substitution is the integral version of the Chain Rule.
   SOLUTION: True. We showed it in class, and gives you some good insight into when to use it.

9. Does the integral converge or diverge? If it converges, evaluate it.

   (a) $\displaystyle\int_0^\infty t\mathrm{e}^{-st}\,dt$
   (s is a constant- state any conditions on s for the integral to converge.)
   SOLUTION: First we'll take care of the integration. Use integration by parts, we get the following (I've put it into a single fraction, but that is not necessary):

$$\begin{array}{c|c|c} + & t & \mathrm{e}^{-st} \\ - & 1 & (-1/s)\mathrm{e}^{-st} \\ + & 0 & (1/s^2)\mathrm{e}^{-st} \end{array} \quad \Rightarrow \quad -\left.\frac{(st+1)\mathrm{e}^{-st}}{s^2}\right|_0^T = \lim_{T\to\infty} -\frac{(sT+1)\mathrm{e}^{-sT}}{s^2} + \frac{1}{s^2}$$

   For the limit, we can factor out the s^2 (it's constant with respect to T), and we get a fraction on which we can use l'Hospital's rule:

$$\frac{-1}{s^2}\lim_{T\to\infty}\frac{sT+1}{\mathrm{e}^{sT}} = \frac{-1}{s^2}\lim_{T\to\infty}\frac{s}{s\mathrm{e}^{sT}} = 0$$

   The previous steps were valid as long as $s>0$ (otherwise, e^{-sT} would diverge to $-\infty$). Overall then, the integral converges to $1/s^2$.

(b) $\displaystyle\int_1^4 \frac{dx}{\sqrt{x-1}}$

SOLUTION: Rewriting the integrand as $(x-1)^{-1/2}$, we see that the antiderivative is $2(x-1)^{1/2} = 2\sqrt{x-1}$. Therefore,

$$\int_1^4 \frac{dx}{\sqrt{x-1}} = \lim_{t\to 1^+} 2\sqrt{x-1}\Big|_t^4 = \lim_{t\to 1^+}\left(2\sqrt{3} - 2\sqrt{t-1}\right) = 2\sqrt{3}$$

(c) $\displaystyle\int_3^\infty \frac{\ln(x)}{x}\,dx$

SOLUTION: The integral diverges. We can use the comparison test with $1/x$. That is, since $1 < \ln(x)$ for $x > e$, then for $x > 3$, we have:

$$\frac{1}{x} < \frac{\ln x}{x}$$

and $\int_3^\infty 1/x\,dx$ diverges.

ALTERNATIVE SOLUTION: You could perform the integration, and show that the limit diverges. In this case, if we take care of the integrand first:

$$\int \frac{\ln(x)}{x}\,dx \qquad \begin{array}{c} u = \ln(x) \\ du = (1/x)\,dx \end{array} \qquad \int u\,du = \frac{1}{2}u^2 = \frac{1}{2}(\ln(x))^2$$

Now we can take the limit:

$$\int_3^\infty \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty} \frac{1}{2}(\ln(x))^2\Big|_3^t$$

This limit diverges to infinity.

(d) $\displaystyle\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$

SOLUTION: Rewrite the integral using a convenient number:

$$\int_{-\infty}^1 \frac{x}{x^2+1}\,dx + \int_1^\infty \frac{x}{x^2+1}\,dx$$

You might guess that these integrals will diverge, since the function is very similar to $1/x$ for large values (either positive or negative) of x.

Formally, we can use the limit test. Since $\int_1^\infty 1/x\,dx$ diverges, and

$$\lim_{x\to\infty} \frac{\frac{x}{x^2+1}}{1/x} = \lim_{x\to\infty}\frac{x^2}{x^2+1} = 1$$

then

$$\int_1^\infty \frac{x}{x^2+1}\,dx$$

will diverge, and therefore so does $\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$.

10. Evaluate using any method, unless specified below:

(a)

$$\int \frac{4\,dx}{(4+x^2)^{3/2}}$$

SOLUTION: Trig substitution is the most direct choice.
Let $x = 2\tan(\theta)$. Then

$$4 + x^2 = 4 + 4\tan^2(\theta) = 4\sec^2(\theta) \qquad \text{and} \qquad dx = 2\sec^2(\theta)\,d\theta$$

Substituting these in, we get:

$$\int \frac{8\sec^2(\theta)\,d\theta}{8\sec^3(\theta)} = \int \cos(\theta)\,d\theta = \sin(\theta) + C$$

Use the reference triangle to convert this answer back to x:

$$\frac{x}{\sqrt{4+x^2}} + C$$

(b) $\int \tan^3(x)\sec^2(x)\,dx$

SOLUTION: This is a trig integral- Try to reserve something to pull off a u, du substitution. In this case, reserve $\sec^2(x)$ so that $u = \tan(x)$ and $du = \sec^2(x)\,dx$, and the integral becomes $\int u^3\,du$.

$$\frac{1}{4}\tan^4(x) + C$$

(c) $\displaystyle\int \frac{3x+2}{x^2+6x+8}\,dx = \int \frac{3x+2}{(x+2)(x+4)}\,dx$

SOLUTION: Since the denominator factors, use partial fractions. Here is the final answer:

$$= \int \frac{5}{x+4} - \frac{2}{x+2}\,dx = 5\ln|x+4| - 2\ln|x+2| + C$$

(d) $\displaystyle\int \frac{t^2\cos(t^3-2)}{\sin^2(t^3-2)}\,dt$

SOLUTION: Look for the u, du substitution first. In this case, we do have what we need, if we let $u = \sin(t^3-2)$. Then the integral becomes

$$\frac{1}{3}\int u^{-2}\,du = -\frac{1}{3}\csc(t^3-2) + C$$

(e) $\int \cos^5(x)\sqrt{\sin(x)}\,dx$

SOLUTION: Look for a substitution first. Looks like we can reserve one of the cosines for the du term, and make $u = \sin(x)$:

$$\int \cos^4(x)\sqrt{\sin(x)}\,[\cos(x)\,dx] = \int (1-\sin^2(x))^2\sqrt{\sin(x)}\,[\cos(x)\,dx] =$$

$$\int (1-u^2)^2\sqrt{u}\,du = \int u^{1/2} - 2u^{5/2} + u^{9/2}\,du = \frac{2}{3}u^{3/2} - \frac{4}{7}u^{7/2} + \frac{2}{11}u^{11/2}$$

To finish up the problem, back substitute the x.

(f) $\int \frac{x}{x^2+4}\,dx$
SOLUTION: Straight u, du substitution: $\frac{1}{2}\ln|x^2+4|+C$.

(g) $\int \frac{dx}{\sqrt{1-6x-x^2}}$
SOLUTION: We'll need to complete the square in the denominator, then probably do a trig substitution. To complete the square, notice that

$$1-6x-x^2 = 1-(x^2+6x+\quad) = 10-(x+3)^2 = \sqrt{10}^2-(x+3)^2$$

I can make the substitution: $x+3=\sqrt{10}\sin(\theta)$ so that the denominator becomes:

$$\sqrt{10-10\sin^2(\theta)} = \sqrt{10}\cos(\theta)$$

and don't forget the dx term: $dx=\sqrt{10}\cos(\theta)\,d\theta$:

$$\int \frac{dx}{\sqrt{1-6x-x^2}} = \int \frac{\sqrt{10}\cos(\theta)\,d\theta}{\sqrt{10}\cos(\theta)} = \theta + C$$

Convert back to x to get

$$\sin^{-1}\left(\frac{x+3}{\sqrt{10}}\right)+C$$

(h) $\int \frac{x-1}{x^2+3}\,dx$
SOLUTION: It might be easiest to separate these into two integrals, or you could do a trig substitution. Separating we get:

$$\int \frac{x-1}{x^2+3}\,dx = \int \frac{x}{x^2+3}\,dx - \int \frac{1}{x^2+3}\,dx$$

The first integral is set up for u, du substitution. For the second integral, factor 3 from the denominator so that we can do a different u, du substitution:

$$\int \frac{1}{x^2+3}\,dx = \frac{1}{3}\int \frac{dx}{\left(\frac{x}{\sqrt{3}}\right)^2+1} = \frac{1}{\sqrt{3}}\int \frac{1}{u^2+1}\,du = \frac{1}{\sqrt{3}}\tan^{-1}(u)$$

Put the two together: $\frac{1}{2}\ln|x^2+3| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+C$

(i) $\int \sin^2(3t)\,dt$
SOLUTION: Use the half angle identity:

$$\int \sin^2(3t)\,dt = \frac{1}{2}\int 1-\cos(6t)\,dt = \frac{1}{2}t - \frac{1}{12}\sin(6t)+C$$

(j) $\int \frac{3x-2}{(x^2+2)^2}\,dx$
SOLUTION: We could break this into two, then use u, du substitution on one and trig substitution on the other, or we can just go for the trig substitution gusto from the start!

Let $x = \sqrt{2}\tan(\theta)$ and make the necessary substitutions to get:

$$\int \frac{3x-2}{(x^2+2)^2} = \int \frac{(3\sqrt{2}\tan(\theta)-2)(\sqrt{2}\sec^2(\theta))}{4\sec^4(\theta)}\,d\theta = \frac{\sqrt{2}}{4}\int \frac{(3\sqrt{2}\tan(\theta)-2)}{\sec^2(\theta)}d\theta$$

Continuing to simplify,

$$\frac{3}{2}\int \sin(\theta)\cos(\theta)\,d\theta - \frac{\sqrt{2}}{2}\int \cos^2(\theta)\,d\theta = \frac{3}{2}\int \sin(\theta)\cos(\theta)\,d\theta - \frac{\sqrt{2}}{4}\int (1+\cos(2\theta))\,d\theta$$

These can now each be evaluated to get:

$$\frac{3}{4}\sin^2(\theta) - \frac{\sqrt{2}}{4}\theta - \frac{\sqrt{2}}{8}\sin(2\theta) = \frac{3}{4}\sin^2(\theta) - \frac{\sqrt{2}}{4}\theta - \frac{\sqrt{2}}{4}\sin(\theta)\cos(\theta)$$

Finally, back substitute x using a triangle (which is why we converted $\sin(2\theta)$ in the previous answer). Unsimplified, the answer is:

$$\frac{3}{4}\left(\frac{x}{\sqrt{x^2+2}}\right)^2 - \frac{\sqrt{2}}{4}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{\sqrt{2}}{4}\frac{x}{\sqrt{x^2+2}}\cdot\frac{\sqrt{2}}{\sqrt{x^2+2}} + C$$

NOTE: If you evaluate $\int \sin(\theta)\cos(\theta)\,d\theta = -\frac{1}{2}\cos^2(\theta)$, you get a slightly different answer...

(k) $\int \sin^{-1}(x)\,dx$

Use integration by parts

$$\begin{array}{ccc} + & \sin^{-1}(x) & 1 \\ - & \frac{1}{\sqrt{1-x^2}} & x \end{array} \quad \Rightarrow \quad x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

Let $u = 1-x^2$, $du = -2x\,dx$ to finish: $x\sin^{-1}(x) + \sqrt{1-x^2} + C$

(l) $\int x^3\sqrt{x^2+4}\,dx$

Substitution: $u = x^2+4$, $du = 2x\,dx$, and $x^2 = u-4$. Then

$$\int x^3\sqrt{x^2+4}\,dx = \frac{1}{2}\int (u-4)u^{1/2}\,du = \frac{1}{2}\int u^{3/2} - 4u^{1/2}\,du$$

(and continue...)

$$\frac{1}{5}(x^2+4)^{5/2} - \frac{4}{3}(x^2+4)^{3/2} + C$$

(m) $\int \sqrt{2x-x^2}\,dx$

Complete the square first: $\int \sqrt{-(x^2-2x+1)+1}\,dx = \int \sqrt{1-(x-1)^2}\,dx$ Use a trig substitution: $\sin(\theta) = x-1$ and $\cos(\theta)\,d\theta = dx$. The integral becomes the following, which we can evaluate using either the half angle formulas or your table of formulas:

$$\int \cos^2(\theta)\,d\theta = \frac{1}{2}\cos(\theta)\sin(\theta) + \frac{1}{2}\theta$$

Use the reference triangle to convert back to x:

$$\frac{1}{2}(\sin^{-1}(x-1) + (x-1)\sqrt{2x-x^2}) + C$$

(n) $\int \sqrt{t}\ln(t)\,dt$

Integration by parts:

$$\begin{array}{ccc} + & \ln(t) & \sqrt{t} \\ - & 1/t & \frac{2}{3}t^{3/2} \end{array} \qquad \Rightarrow \qquad \frac{2}{3}t^{3/2}\ln(t) - \frac{2}{3}\int t^{1/2}\,dt = \frac{2}{3}t^{3/2}\ln(t) - \frac{4}{9}t^{3/2} + C$$

(o) $\displaystyle\int \frac{3x-1}{(x+2)(x-3)}\,dx$

By partial fractions,

$$\frac{3x-1}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3} \qquad \Rightarrow \qquad 3x-1 = A(x-3) + B(x+2)$$

Substitute $x = 3$ to get $A = 7/5$ and substitute $x = -2$ to get $B = 8/5$. Then the integral becomes:

$$\int \frac{3x-1}{(x+2)(x-3)}\,dx = \frac{7}{5}\int \frac{1}{x+2}\,dx + \frac{8}{5}\int \frac{1}{x-3}\,dx = \frac{7}{5}\ln|x+2| + \frac{8}{5}\ln|x-3| + C$$

(p) $\int \ln(y^2+9)\,dy$

SOLUTION: Just like the regular log, we can integrate by parts

$$\begin{array}{c|c|c} + & \ln(y^2+9) & 1 \\ \hline - & \frac{2y}{y^2+9} & y \end{array} \qquad \Rightarrow \qquad y\ln(y^2+9) - 2\int \frac{y^2}{y^2+9}\,dy$$

For the integral in y, we can use trig substitution: $y = 3\tan(\theta)$ so that $y^2 + 9 = 9(\tan^2(\theta) + 1) = 9\sec^2(\theta)$ and $dy = 3\sec^2(\theta)$:

$$\int \frac{y^2}{y^2+9}\,dy = \int \frac{9\tan^2(\theta)(3\sec^2(\theta))}{9\sec^2(\theta)}\,d\theta = 3\int \tan^2(\theta)\,d\theta$$

Now, use the formulas that will be handed out (about half way down the page) to get that

$$3\int \tan^2(\theta)d\theta = 3\left(\tan(\theta) - \theta\right)$$

Convert back to y so that:

$$-2\int \frac{y^2}{y^2+9}\,dy = -6\cdot\frac{y}{3} + 6\tan^{-1}\left(\frac{y}{3}\right)$$

Put it all together:

$$y\ln(y^2+9) - 2y + 6\tan^{-1}(y/3) + C$$

(q) $\displaystyle\int \frac{\sin^3(x)}{\cos^4(x)}\,dx$

Retain one $\sin(x)$ to go with dx, and set up the substitution $u = \cos(x)$ $du = -\sin(x)\,dx$:

$$-\int (1-u^2)u^{-4}\,du = -\int u^{-4} - u^{-2}\,du = \frac{1}{3}\sec^3(x) - \sec(x) + C$$

(r) $\int e^{-x}\sin(2x)\,dx$

Integrate by parts twice to get the same integral on both sides,

$$\begin{array}{ccc} + & \sin(2x) & e^{-x} \\ - & 2\cos(2x) & -e^{-x} \\ + & -4\sin(2x) & e^{-x} \end{array}$$

Therefore, we have:

$$\int e^{-x}\sin(2x)\,dx = -e^{-x}(\sin(2x) + 2\cos(x)) - 4\int e^{-x}\sin(2x)\,dx$$

and

$$\int e^{-x}\sin(2x)\,dx = -\frac{1}{5}e^{-x}(\sin(2x) + 2\cos(x)) + C$$

(s) $\int \frac{w}{\sqrt{w+5}}\,dw$

SOLUTION: After some trial and error, we might take

$$u = \sqrt{w+5}$$

We'll need to solve this for w and dw to make the substitution:

$$w = u^2 - 5 \quad \Rightarrow \quad dw = 2u\,du$$

Therefore,

$$\int \frac{w}{\sqrt{w+5}}\,dw = \int \frac{(u^2-5)2u\,du}{u} = \frac{2}{3}u^3 - 10u + C =$$

$$\frac{2}{3}(w+5)^{3/2} - 10(w+5)^{1/2} + C$$

(t) $\int y^2 e^{-3y}\,dy$

SOLUTION: Integration by parts using a table

$$\begin{array}{c|c|c} + & y^2 & e^{-3y} \\ - & 2y & (-1/3)e^{-3y} \\ + & 2 & (1/9)e^{-3y} \\ - & 0 & (-1/27)e^{-3y} \end{array}$$

Then just write out the answer. Notice that we can factor out $-e^{-3y}$ to get:

$$-e^{-3y}\left(\frac{1}{3}y^2 + \frac{2}{9}y + \frac{2}{27}\right) + C$$