Quiz 2 Solutions

Part of this quiz was to see if you could follow the directions!

- Solutions are written neatly, clearly and completely using your own paper (up to 10 pts)
- Quiz is stapled (up to 10 pts).
- Quiz turned in on time (up to 10 pts).

Here are the remaining solutions (10 points per problem, so 70 points total):

1. Find the limit of the given sequence $\{a_n\}$, if it exists:

   (a) $a_n = n\sin(\pi/n)$ (Hint: The technique on page 305 may help.)
   SOLUTION: The technique on p. 305 is to use l'Hospital's rule:

$$\lim_{n\to\infty} n\sin(\pi/n) = \lim_{n\to\infty} \frac{\sin(\pi/n)}{1/n} = \lim_{n\to\infty} \frac{\cos(\pi/n)(-\pi/n^2)}{-1/n^2} = \pi\cos(0) = \pi$$

   (b) $a_n = \dfrac{(2n-1)!}{(2n+1)!}$
   SOLUTION: First simplify, then take the limit.

$$a_n = \frac{1\cdot 2\cdot 3\cdots(2n-1)}{1\cdot 2\cdot 3\cdots(2n-1)(2n)(2n+1)} = \frac{1}{2n(2n+1)}$$

   Now, the limit as $n\to\infty$ for this quantity is zero.

2. Test the series for convergence or divergence. If the series converges, say whether it is absolute or conditional. Be specific about your reasons!

   (a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{10^n}{n!}$
   SOLUTION: We'll try the Ratio Test since the terms involve $n!$.

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{10^{n+1}}{(n+1)!}\cdot\frac{n!}{10^n} = \lim_{n\to\infty}\frac{10}{n+1} = 0$$

   Therefore, the series converges absolutely by the Ratio Test.

   (b) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{2\cdot 5\cdot 8\cdots(3n+2)}$
   SOLUTION: Another Ratio Test, since we have factorials galore!

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{(n+1)!}{2\cdot 5\cdot 8\cdots(3n+2)(3n+5)}\cdot\frac{2\cdot 5\cdot 8\cdots(3n+2)}{n!}$$

$$\lim_{n\to\infty}\frac{n+1}{3n+5} = \frac{1}{3}$$

   Therefore, by the Ratio Test, the series converges absolutely.

   (c) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}-1}$
   SOLUTION: First, if we were to check for absolute convergence, the series would be

$$\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}-1}$$

   Using the Limit Comparison test with $1/\sqrt{n}$ (which represents a divergent $p-$series), the limit is

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}-1}\frac{\sqrt{n}}{1} = \lim_{n\to\infty}\frac{\sqrt{n}/\sqrt{n}}{(\sqrt{n}-1)/\sqrt{n}} = \lim_{n\to\infty}\frac{1}{1-\frac{1}{\sqrt{n}}} = 1$$

Therefore, by the Limit Comparison test, the series $\sum \frac{1}{\sqrt{n}-1}$ diverges.

However, we'll use the Alternating Series test to show that the series converges (conditionally). Let $b_n = \frac{1}{\sqrt{n}-1}$.

Since $\sqrt{n+1} - 1 > \sqrt{n} - 1$, then $b_{n+1} < b_n$ (therefore, the sequence of b_n is decreasing).

And, the limit of b_n is zero, as shown below:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}-1} = 0$$

Therefore, the series converges by the Alternating Series Test.

Now, since the series did not converge absolutely, but does converge by the Alternating Series Test, the series converges conditionally.

3. Find all value(s) of x for which the series converges. Find the sum of the series for those value(s) of x: $\displaystyle\sum_{n=1}^{\infty} \frac{(x+3)^n}{2^n}$

SOLUTION: Comparing this to a Geometric series, the ratio r is

$$r = \frac{x+3}{2}$$

and the series converges (absolutely) if $|r| < 1$, or

$$\frac{|x+3|}{2} < 1 \quad \Rightarrow \quad |x+3| < 2 \quad \Rightarrow \quad -2 < x+3 < 2 \quad \Rightarrow \quad -5 < x < -1$$

and the sum of the series is:

$$\frac{r}{1-r} = \frac{\frac{x+3}{2}}{1 - \frac{x+3}{2}} = \frac{x+3}{2} \cdot \frac{2}{2-x-3} = \frac{x+3}{-x-1} = -\frac{x+3}{x+1}$$

4. How many terms of the sum do we need in order to find the sum to the indicated accuracy?

$$\sum_{n=1}^{\infty} (-1)^{n-1} n e^{-n} \qquad (|\text{error}| < 0.01)$$

SOLUTION: Let $b_n = ne^{-n}$ and start computing b_n until we get a number less than the error. Here are the numbers I get

n	b_n
1	0.3679
2	0.2707
3	0.1494
4	0.0733
5	0.0337
6	0.0149
7	0.0064

$\Rightarrow \quad |R_n| = 0.01 < b_{n+1}$

We see that taking 6 terms of the sum will mean that the remainder (as an approximation to the sum) will be within 0.01 of the actual amount (in fact, it will be within 0.0064 of the actual amount).

If you're curious, using 6 terms actually gives an error of 0.004487, but that is beyond the scope of the quiz.