Review Solutions: Chapter 11

1. What does it mean to say that a series converges?

   SOLUTION: We define the n^{th} partial sum s_n as follows:

$$S_1 = a_1 \qquad S_2 = a_1 + a_2 \qquad S_3 = a_1 + a_2 + a_3 \quad \cdots \quad S_n = \sum_{k=1}^{n} a_n$$

   The partial sums S_n form a sequence (of numbers). The (infinite) series is said to *converge* to a sum S if and only if the limit of the partial sums is S. That is, if the limit

$$\lim_{n\to\infty} S_n = S$$

   then the series is said to converge, and the sum is said to be equal to S:

$$\sum_{n=1}^{\infty} a_n = S$$

2. Does the given sequence or series converge or diverge?

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$

   SOLUTION: Using the dominating terms, this looks a lot like $\sum \frac{1}{n}$, so we use the limit comparison (note that both series, the one given and the template, have all positive terms)

$$\lim_{n\to\infty} \frac{\frac{1}{n-\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n-\sqrt{n}} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{\sqrt{n}}} = 1$$

   Therefore, both series diverge together (by the limit comparison test).

   (b) $\left\{\frac{n}{1+\sqrt{n}}\right\}$

   SOLUTION: Take the limit; You can use L'Hospital's rule if you like. To be precise, we ought to change notation to x (since you cannot formally take the derivative of a discrete sequence):

$$\lim_{x\to\infty} \frac{x}{1+\sqrt{x}} = \lim_{x\to\infty} \frac{1}{1/2\sqrt{x}} = \lim_{x\to\infty} 2\sqrt{x} = \infty$$

   Therefore, the sequence diverges.

   (c) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+1}{n^3-1}$

   SOLUTION: This looks again like the harmonic series (which diverges). Use the limit comparison with $\sum 1/n$:

$$\lim_{n\to\infty} \frac{\frac{n^2-1}{n^3-1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3-n}{n^3-1} = 1$$

   Therefore, both series diverge together by the limit comparison test.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{5-2\sqrt{n}}{n^3}$

We can temporarily break this apart to see if the pieces converge:

$$\sum_{n=1}^{\infty} \frac{5-2\sqrt{n}}{n^3} = \sum_{n=1}^{\infty} \frac{5}{n^3} - 2\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3}$$

Both of these are $p-$series, the first with $p = 3$, the second with $p = \frac{5}{2}$, therefore they converge separately, and so the sum also converges.

A good thing to know: The sum of two convergent series is a convergent series.

(e) $\displaystyle\sum_{n=1}^{\infty} (-6)^{n-1} 5^{1-n}$

SOLUTION: First, let's rewrite the terms of the sum:

$$(-6)^{n-1} 5^{1-n} = \frac{(-6)^{n-1}}{5^{n-1}} = \left(\frac{-6}{5}\right)^{n-1}$$

so that this is a geometric series with $r = \frac{-6}{5}$. Since $|r| > 1$, this series diverges.

(f) $\left\{ \frac{n!}{(n+2)!} \right\}$

SOLUTION: We first simplify:

$$\frac{n!}{(n+2)!} = \frac{1}{(n+1)(n+2)}$$

so the limit as $n \to \infty$ is 0.

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{5^n n!}$

SOLUTION: With factorials and powers, use the Ratio Test. Because all terms are always positive, we can drop the absolute value signs (if it converges, it would be absolute convergence). Before taking the limit, we can simplify algebraically:

$$\frac{a_{n+1}}{a_n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{5^{n+1}(n+1)!} \cdot \frac{5^n n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{2n+1}{5(n+1)} = \frac{2n+1}{5n+5}$$

Now, take the limit:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{5+\frac{5}{n}} = \frac{2}{5}$$

Since the limit is less than 1, the series converges (absolutely) by the Ratio Test.

(h) $\displaystyle\sum_{n=2}^{\infty} \frac{3^n + 2^n}{6^n}$

A sum of (convergent) geometric series is also convergent. In fact, we can find the sum to which the series will converge:

$$\sum_{n=2}^{\infty} \frac{3^n + 2^n}{6^n} = \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n = \frac{(1/2)^2}{1-(1/2)} + \frac{(1/3)^2}{1-(1/3)} = \frac{2}{3}$$

(i) $\left\{\sin\left(\frac{n\pi}{2}\right)\right\}$

SOLUTION: Write out the first few terms of the sequence:

$$1, 0, -1, 0, 1, 0, -1, \ldots$$

so the sequence diverges.

(j) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$

SOLUTION: We see the terms go to zero like $\frac{1}{n^3}$ (that would be a convergent p series). Therefore, use the limit comparison test:

$$\lim_{n\to\infty} \frac{n^3}{n(n+1)(n+2)} = 1$$

so the series converges by the limit comparison test.

NOTE: Did you try to use the Ratio Test? The Ratio (and Root) tests always give an inconclusive answer for any $p-$series.

(k) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n\sqrt{n}}$

SOLUTION: First, do the terms go to zero? The maximum value of the sine function is 1, and all terms of the sum are positive, so:

$$\frac{\sin^2(n)}{n^{3/2}} \leq \frac{1}{n^{3/2}}$$

so the terms do go to zero. Actually, we've also done a direct comparison with the $p-$series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which converges.

(l) $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{n^2 9^n}$

SOLUTION: Ratio Test (note that the negative sign is meaningless since $(-1)^{2n} = 1$). Start with some algebra to simplify before taking the limit:

$$\frac{5^{2n+2}}{(n+1)^2 9^{n+1}} \cdot \frac{n^2 9^n}{5^{2n}} = \left(\frac{n}{n+1}\right)^2 \cdot \frac{5^{2n}5^2}{5^{2n}} \cdot \frac{9^n}{9^{n+1}} = \left(\frac{n}{n+1}\right)^2 \cdot \frac{25}{9}$$

The limit as $n \to \infty$ is $25/9$:

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 \cdot \frac{25}{9} = \left(\lim_{n\to\infty} \frac{n}{n+1}\right)^2 \cdot \frac{25}{9} = \frac{25}{9} > 1$$

Therefore, the series diverges by the Ratio Test.

(m) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)2^n}$

SOLUTION: It looks like the terms are going to zero like $\frac{1}{2^n}$, so let's compare it to $\sum(1/2)^n$, which is a convergent geometric series.

$$\frac{n}{n+1} \cdot \frac{1}{2^n} \leq \frac{1}{2^n}$$

So the series converges by a direct comparison. The Ratio Test would work well here, too.

3. Find the sum of the series

NOTE: We only know two ways of finding the sum for a convergent series- Either by using the Geometric Series or by using the Taylor Series of a template series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{2^{2n}}$

SOLUTION: Do some algebra first. This should look like a geometric series(?)

$$\frac{(-3)^{n-1}}{2^{2n}} = \frac{(-3)^n(-3)^{-1}}{(2^2)^n} = -\frac{1}{3}\cdot\left(-\frac{3}{4}\right)^n$$

Now, this is a convergent series with $a = -1/3$ and $r = -3/4$. The sum is:

$$\frac{(-1/3)(-3/4)}{1+\frac{3}{4}} = \frac{1}{4}\cdot\frac{4}{7} = \frac{1}{7}$$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{(x-3)^{2n}}{3^n}$

This is a geometric series with $r = \frac{(x-3)^2}{3}$. Putting it into the formula for the sum,

$$\frac{\left(\frac{(x-3)^2}{3}\right)^2}{1-\frac{(x-3)^2}{3}} = \frac{(x-3)^4}{9}\cdot\frac{3}{3-(x-3)^2} = \frac{3(x-3)^4}{3-(x-3)^2}$$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\pi^{2n}}{3^{2n}(2n!)}$

SOLUTION: The series is a cosine series (see the even powers of x?). We might do a little algebra first:

$$\sum_{n=0}^{\infty} \frac{(-1)^n\pi^{2n}}{3^{2n}(2n!)} \qquad\Rightarrow\qquad \sum_{n=0}^{\infty} \frac{(-1)^n(\pi/3)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

4. (a) $\displaystyle\sum \frac{n!x^n}{1\cdot 3\cdot 5\cdots(2n-1)}$

SOLUTION:

Use the Ratio Test (and remember to use the absolute value signs!). First a little algebra:

$$\frac{(n+1)!|x|^{n+1}}{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!|x|^n} = \frac{n+1}{2n+1}|x|$$

Now take the limit and apply the Ratio test:

$$|x|\lim_{n\to\infty}\frac{n+1}{2n+1} = \frac{|x|}{2} < 1 \quad\Rightarrow\quad |x| < 2$$

Therefore, the radius of convergence is 2.

(b) $\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n^2 5^n}$

SOLUTION: Use the Ratio test- First simplify.

$$\frac{|x|^{n+1}}{(n+1)^2 5^{n+1}} \cdot \frac{n^2 5^n}{|x|^n} = \left(\frac{n}{n+1}\right)^2 \cdot \frac{|x|}{5}$$

Now take the limit and apply the test:

$$\frac{|x|}{5} \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = \frac{|x|}{5}\left(\lim_{n\to\infty}\frac{n}{n+1}\right)^2 = \frac{|x|}{5} < 1 \quad \Rightarrow \quad |x| < 5$$

The radius of convergence is 5. When we test $x = -5$ and $x = 5$, we get convergent p series ($\sum 1/n^2$ and $\sum(-1)^n/n^2$, respectively. Therefore, the interval of convergence is

$$[-5, 5]$$

(c) $\sum_{n=0}^{\infty} \frac{2^n(x-3)}{\sqrt{n+3}}$

SOLUTION: Another Ratio Test... In this case, the series is centered at $x = 3$, so we'll have an exciting change of pace in calculating the interval of convergence! Here we go- As usual, do the algebra first:

$$\frac{2^{n+1}|x-3|^{n+1}}{\sqrt{n+4}} \frac{\sqrt{n+3}}{2^n|x-3|^n} = 2|x-3|\sqrt{\frac{n+3}{n+4}}$$

The limit can be brought under the radical sign since the square root is a continuous function:

$$2|x-3| \lim_{n\to\infty}\sqrt{\frac{n+3}{n+4}} = 2|x-3|\sqrt{\lim_{n\to\infty}\frac{n+3}{n+4}} = 2|x-3|$$

To apply the Ratio test, if $2|x-3| < 1$, the series will converge absolutely. Therefore, the radius of convergence is $1/2$ and to find the interval of convergence, we test the endpoints:

$$-\frac{1}{2} < x - 3 < \frac{1}{2} \quad \Rightarrow \quad \frac{5}{2} < x < \frac{7}{2}$$

If we put in $x = 5/2$, the series becomes

$$\sum \frac{2^n \cdot \left(\frac{-1}{2}\right)^n}{\sqrt{n+3}} = \sum \frac{(-1)^n}{\sqrt{n+3}}$$

This will converge by the Alternating Series Test (diverges absolutely since it is similar to a divergent p-series): (i) It is alternating. (ii) It is decreasing: $\sqrt{n+4} > \sqrt{n+3}$, so $1/\sqrt{n+4} < 1/\sqrt{n+3}$. (iii) The terms go to zero.

If we put in $x = 7/2$, we get something similar to a divergent p series, which diverges:

$$\sum \frac{1}{\sqrt{n+3}}$$

We could show it by the limit comparison test with $1/\sqrt{n}$.

Summary: The interval is $[5/2, 7/2)$

5. Use a series to evaluate the following limit: $\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$

   SOLUTION: Use our template series for the sine function- In fact, write it out:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

   so

$$\sin(x) - x = -\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

   and

$$\frac{\sin(x) - x}{x^3} = -\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \cdots$$

   Evaluate this series at $x = 0$ (because we want the limit as $x \to 0$) to get the answer of $-1/6$.

   Optional: We can verify our answer using L'Hospital's rule applied several times:

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3} = \lim_{x\to 0} \frac{\cos(x) - 1}{3x^2} = \lim_{x\to 0} \frac{-\sin(x)}{6x} = \lim_{x\to 0} \frac{-\cos(x)}{6} = -\frac{1}{6}$$

6. Use a known template to find a series for the following:

   (a) $\frac{x^2}{1+x}$

   SOLUTION: This looks kinda like the sum of a geo series:

$$\frac{x^2}{1+x} = x^2 \cdot \frac{1}{1+x} = x^2 \cdot \frac{1}{1-(-x)} = x^2 \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n x^{n+2}$$

   (b) 10^x

   SOLUTION: The only template that comes close would be the exponential function, e^x. We recall the algebra:

$$10 = \mathrm{e}^{\ln(10)}$$

   so that

$$10^x = \left(\mathrm{e}^{\ln(10)}\right)^x = \mathrm{e}^{(\ln(10)x)}$$

   so in the series for the exponential function, replace x by $x\ln(10)$:

$$10^x = \sum_{n=0}^{\infty} \frac{(x\ln(10))^n}{n!} = \sum_{n=0}^{\infty} \frac{(\ln(10))^n x^n}{n!}$$

   (c) $x\mathrm{e}^{2x}$

   SOLUTION: Start with the series for e^x, and substitute $2x$ in where we see an x. To get the series for $x\mathrm{e}^{2x}$, multiply the series by x:

$$\mathrm{e}^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} \quad \Rightarrow \quad x\mathrm{e}^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n!}$$

7. Find the Taylor series for $f(x)$ centered at the given base point:

(a) $x^4 - 3x^2 + 1$, at $x = 1$

Set up the table:

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^4 - 3x^2 + 1$	-1
1	$4x^3 - 6x$	-2
2	$12x^2 - 6$	6
3	$24x$	24
4	24	24
5	0	0

$$\Rightarrow \quad f(x) = -1 - 2(x-1) + \frac{6}{2!}(x-1)^2 + \frac{24}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4$$

The other terms of the sum are zero.

Optional note: If you expand your Taylor series and simplify, you should get the original polynomial.

(b) $1/\sqrt{x}$ at $x = 9$ (just get the first four non-zero terms of the power series).

n	$f^{(n)}(x)$	$f^{(n)}(9)$
0	$x^{-1/2}$	$1/3$
1	$(-1/2)x^{-3/2}$	$-1/54$
2	$(3/4)x^{-5/2}$	$1/324$
3	$(-15/8)x^{-7/2}$	$-5/5832$

$$\Rightarrow \quad \frac{1}{\sqrt{x}} \approx \frac{1}{3} - \frac{1}{54}(x-9) + \frac{1}{648}(x-9)^2 - \frac{5}{34992}(x-9)^3$$

(c) $1/x^2$ at $x = 1$. In this case, find a pattern for the n^{th} coefficient so that you can write the general series. Using this answer, find the radius of convergence.

SOLUTION: Build a table

n	$f^{(n)}(x)$	$f^{(n)}(9)$
0	x^{-2}	1
1	$(-2)x^{-3}$	-2
2	$(3)(2)x^{-4}$	$3!$
3	$-(4)(3)(2)x^{-5}$	$-4!$
$\vdots$	$\vdots$	$\vdots$
n		$(-1)^n(n+1)!$

$$\begin{aligned} f(x) &= \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{n!}(x-1)^n \\ &= \textstyle\sum_{n=0}^{\infty} (-1)(n+1)(x-1)^n \end{aligned}$$

With the Ratio Test we get (after the algebra):

$$|x-1| \lim_{n\to\infty} \frac{n+2}{n+1} = |x-1| < 1$$

Therefore, the radius of convergence is 1. *NOTE:* We could have anticipated that, since $1/x^2$ has a vertical asymptote at $x = 0$, and our base point is $x = 1$.

8. True or False, and give a short reason:

(a) If $\lim_{n\to\infty} a_n = 0$, then the series $\sum a_n$ is convergent.

FALSE. For example, $1/n$ goes to zero, but the series diverges.

(b) If $\sum c_n 6^n$ is convergent, so is $\sum c_n(-2)^n$.

TRUE. This is an interesting question, since we don't know the base point- but we can take the base point to be any value!

For the convergent series, $x - a = 6$, so $x = 6 + a$. Therefore, x could be as much as 6 units from the base point for convergence. To get the quantity -2, we would have:

$$x - a = -2 \quad \Rightarrow \quad x = -2 + a$$

and so in this case, x is only 2 units away from the base point- Therefore, it must converge in this case as well.

(c) The Ratio Test can be used to determine if a $p-$series is convergent.

FALSE: Unfortunately, the Ratio Test always fails for $p-$series. For example, given $1/n^p$, the Ratio Test gives:

$$\lim_{n\to\infty} \frac{n^p}{(n+1)^p} = \left(\lim_{n\to\infty} \frac{n}{n+1}\right)^p = 1^p = 1$$

(d) If $0 \le a_n \le b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

False. If $\sum a_n$ were divergent, we could then conclude that $\sum b_n$ diverges.

(e) $0.9999999\ldots = 1$

True: Let $s = 0.9999999\ldots$. Then

$$\begin{array}{rl} 10s & = 9.999999\ldots \\ -s & = -0.999999\ldots \\ \hline 9s & = 9.000000\ldots \end{array}$$

Therefore, $s = 1$.

(f) If $a_n > 0$ and $\sum a_n$ converges, then $\sum (-1)^n a_n$ converges.

True. Notice that $|(-1)^n a_n| = a_n$ since $a_n > 0$- Therefore, $\sum(-1)^n a_n$ converges absolutely. If a series converges absolutely, it must also converge.

(g) If $f(x) = 2x - x^2 + \frac{1}{3}x^3 - \cdots$ converges for all x, then $f'''(0) = 2$.

True. The term with x^3 gives the third derivative:

$$\frac{f'''(0)}{6} = \frac{1}{3} \quad \Rightarrow \quad f'''(0) = 2$$

(You could just start differentiating as well).

9. Suppose that $\sum_{n=0}^{\infty} c_n(x-1)^n$ converges when $x = 3$ and diverges when $x = -2$. What can be said about the convergence or divergence of the following?

NOTE: There are several ways of solving this, but from what we're given, we know that the series must converge for all x between -1 and 3, and must diverge for all $x \le -2$, and $x > 4$. We don't know what happens at $x = 4$.

(a) $\sum c_n$

In this case, $x = 2$ is in the interval of convergence.

(b) $\sum(-1)^n c_n$

In this case, $x = 0$, which is also in the interval where we know the series converges.

(c) $\sum c_n 3^n$

This point is for $x = 4$. At this value, the series could converge or diverge- We would need more information.

10. Find the sum: $\sum \frac{n(n-1)}{2^n}$ Hint: Use the geometric series and the derivative

SOLUTION: This sum looks like a second derivative, with $r = \frac{1}{2}$. Let's see if we can formalize this by differentiating the geo series twice:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \qquad \frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

The only problem is that our sum has 2^n in the denominator, not 2^{n-2}. But we can fix that by multiplying both sides by of the last expression by x^2:

$$\frac{x^2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^n$$

Now substitute $x = 1/2$ and compute the sum:

$$\frac{1/4}{(1-1/2)^2} = 2$$

11. Find the Maclaurin series for $\ln(x+1)$ and find the radius of convergence.

SOLUTION: Set up a table to find the pattern in the derivatives:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1+x)$	0
1	$(1+x)^{-1}$	1
2	$(-1)(1+x)^{-2}$	-1
3	$(2)(1+x)^{-3}$	2
4	$(-3)(2)(1+x)^{-4}$	$-3!$
5	$(4)(3)(2)(1+x)^{-5}$	$4!$
$\vdots$	$\vdots$	$\vdots$
n		$(-1)^{n+1}(n-1)!$

$$\begin{aligned}\ln(1+x) &= \textstyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!x^n}{n!} \\ &= \textstyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}\end{aligned}$$

We start the index at $n = 1$ since the first entry is different than the others. Now use the Ratio Test to find the radius of convergence:

$$\frac{|x|^{n+1}}{(n+1)}\frac{n}{|x|^n} = \left(\frac{n}{n+1}\right)|x| \to |x| < 1$$

so the radius is 1.

12. Find the sum: $3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \cdots$

SOLUTION: After some analysis, we see that this could be written as:

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} \qquad \text{at } x = 3$$

This is almost the series for e^x- We're missing the first term. Adding it in, we have:

$$1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x$$

Therefore, evaluating everything at $x = 3$, we have:

$$\sum_{n=1}^{\infty} \frac{3^n}{n!} = e^3 - 1$$

13. Let $a_n = \frac{2n}{3n+1}$

(a) Determine whether $\{a_n\}$ is convergent.

SOLUTION: The *sequence* is convergent, and converges to 2/3.

(b) Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

SOLUTION: Since the sequence of terms a_n went to 2/3, the series diverges (by the Test for Divergence).

14. Same as the previous problem, but use $a_n = \dfrac{1 + 2^n}{3^n}$

(a) Determine whether $\{a_n\}$ is convergent.

SOLUTION: The *sequence* is convergent. Divide numerator and denominator by 3^n, and we get

$$\lim_{n\to\infty} \frac{\frac{1}{3^n} + \left(\frac{2}{3}\right)^n}{1} = 0 + 0 = 0$$

(b) Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

SOLUTION: This is a sum of two convergent geometric series, so the sum will be convergent as well

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n} = \sum_{n=1}^{\infty} \left[\left(\frac{1}{3}\right)^n + \left(\frac{2}{3}\right)^n\right] = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

(We could actually find the sum, but that wasn't part of the question)

15. Compute the sum $1 + a + a^2 + a^3 + a^4 + a^5 + a^6 + a^7$.

Hint: Your answer should be in terms of a fraction. You might start with s equal to the expression above.

SOLUTION: Start out by defining $S = 1 + a + a^2 + a^3 + a^4 + a^5 + a^6 + a^7$. Then

$$\begin{array}{rlllllllll} S & = 1 & +a & +a^2 & +a^3 & +a^4 & +a^5 & +a^6 & +a^7 & \\ -aS & = & -a & -a^2 & -a^3 & -a^4 & -a^5 & -a^6 & -a^7 & -a^8 \\ \hline (1-a)S & = 1 & & & & & & & & -a^8 \end{array} \quad \Rightarrow \quad S = \frac{1-a^8}{1-a}$$

16. Explain the difference between absolute and conditional convergence. Which is "better" and why?

    SOLUTION: Absolute convergence is "better"- If a series is absolutely convergent, it behaves more like a finite sum. For example, if a series converges absolutely, then any re-arrangement of the terms of the sum will converge to the same number. Unfortunately, if a series is only conditionally convergent, it is possible to re-arrange the terms of the sum so that the series will converge to *any* real number.