Review Sheet 1 Solutions

1. Prove by induction: $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

   SOLUTION:

   - Prove the first case: If $n = 1$, then does $1^2 = \frac{1\cdot 2\cdot 3}{6}$? Yes.
   - Assume true if $n = k$. That is: $\displaystyle\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$.

     Use the assumption to prove that the statement is true if $n = k+1$. In that case, we want to show that

     $$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

     Here now is the proof:

     $$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

     Get a common denominator and factor out $k+1$:

     $$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{(k+1)(k(2k+1) + 6(k+1))}{6} =$$

     $$\frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

2. State the definition of $\displaystyle\int_a^b f(x)\,dx$.

   The definition uses the limit of the Riemann sum. You might also state the definition using right endpoints, as we did below- It may help on other problems.

   $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\cdot \Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\cdot\frac{b-a}{n}\right)\cdot\frac{b-a}{n}$$

   KEY IDEA: If the definition is asked for, you must use a Riemann sum, and NOT the Fundamental Theorem of Calculus.

3. True or False, and give a short reason:

   (a) The Alternating Series Test is sufficient to show that a series is conditionally convergent.

   FALSE, because if you go directly to the Alternating Series Test, you do not know if the series converges absolutely.

   (b) You can use the Integral Test to show that a series is absolutely convergent.

   TRUE. The Direct Comparison, Limit Comparison, Ratio Test and Integral tests were all used on positive series.

(c) Consider $\sum a_n$. If $\lim_{n\to\infty} a_n = 0$, then the sum is said to converge.

FALSE. For example, $1/n \to 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Although, if the terms do NOT go to zero, then the sum diverges by the Test for Divergence.

(d) The sequence $a_n = 0.1^n$ converges to $\frac{1}{1-0.1}$

FALSE. This is a sequence, not a series. The sequence converges to zero.

4. Set up an integral for the volume of the solid obtained by rotating the region defined by $y = \sqrt{x-1}$, $y = 0$ and $x = 5$ about the $y-$axis.

SOLUTION: We might use shells. In that case, the height of a shell is $\sqrt{x-1}$ and the radius is x. The volume is then:

$$\int_1^5 2\pi x\sqrt{x-1}\,dx$$

ALTERNATE SOLUTION: If we use washers, we would have:

$$\pi\int_0^2 \left((y^2+1)^2 - 5^2\right)\,dy$$

5. Write the area under $y = \sqrt[3]{1+x}$, $1 \le x \le 4$ as the limit of a Riemann sum (use right endpoints)

SOLUTION: The i^{th} right endpoint is: $1 + \frac{3i}{n}$. The height at this endpoint is given by:

$$f\left(1+\frac{3i}{n}\right) = \sqrt[3]{1+1+\frac{3i}{n}} = \sqrt[3]{2+\frac{3i}{n}}$$

Therefore, the integral is:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt[3]{2+\frac{3i}{n}}\cdot\frac{3}{n}$$

Secondly, for the arc length, we need to differentiate y:

$$y' = \frac{1}{3}(1+x)^{-2/3} \quad\Rightarrow\quad (y')^2 = \frac{1}{9}(1+x)^{-4/3}$$

so the integral is:

$$\int_1^4 \sqrt{1+\frac{1}{9(1+x)^{4/3}}}\,dx$$

6. Find the Taylor series for $f(x) = \sqrt{x}$ centered at $a = 9$.

SOLUTION: The first few derivatives are:

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$\sqrt{x}$	2
1	$(1/2)x^{-1/2}$	$(1/4)$
2	$-(1/4)x^{-3/2}$	$-1/32$
3	$(3/8)x^{-5/2}$	$3/256$

(I'll try to make the numbers work out nicely on the exam) Therefore,

$$\sqrt{x} = 2 + \frac{1}{4}(x-4) - \frac{1}{2\cdot 32}(x-4)^2 + \frac{3}{6\cdot 256}(x-4)^3 + \cdots$$

Main Point: Recall the formula for the Taylor series, and be able to compute the first few terms of a Taylor series.

7. Find $\frac{dy}{dx}$, if $y = \int_{\cos(x)}^{5x} cos(t^2)\; dt$

   SOLUTION: The general formula we got using the Fundamental Theorem of Calculus, Part I was:

   $$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$$

   Therefore, in this particular case,

   $$\cos(25x^2)\cdot 5 + \cos\left(\cos^2(x)\right)\cdot\sin(x)$$

8. Let $f(x) = \mathrm{e}^x$ on the interval $[0,2]$. (a) Find the average value of f. (b) Find c such that $f_{\rm avg} = f(c)$.

   SOLUTION: Remember the theorem: If f is continuous on $[a,b]$, then there is a c in $[a,b]$ so that:

   $$f_{avg} = f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

   So, we first compute the average, then we'll find the c:

   $$\frac{1}{2}\int_0^2 \mathrm{e}^x\,dx = \frac{\mathrm{e}^2-1}{2}$$

   so that $c = \ln\left(\frac{\mathrm{e}^2-1}{2}\right) \approx 1.16$

9. Use a template series to find the series for $\int\cos(x^2)\,dx$.

   SOLUTION: The template series is the series for $\cos(x)$. In that formula, substitute x^2 for x, then integrate:

   $$\cos(x^2) = \sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n}}{(2n)!} \quad\Rightarrow\quad \int\cos(x^2)\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+1}}{(4n+1)(2n)!} + C$$

10. Does the series converge (absolute or conditional), or diverge?

    (a) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n/2)}{n^2+4n}$ Note that $\cos(n/2) < 1$ for all n. Therefore, by direct comparison, $\left|\frac{\cos(n/2)}{n^2+4n}\right| < \frac{1}{n^2}$ Thus, the series converges absolutely by the direct comparison test.

    (b) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+1}{5^n}$ Using the ratio test, $\displaystyle\lim_{n\to\infty}\frac{(n+1)^2+1}{5^{n+1}}\cdot\frac{5^n}{n^2+1} = \lim_{n\to\infty}\frac{(n+1)^2+1}{n^2+1}\cdot\frac{1}{5} = \frac{1}{5}$ The series converges absolutely by the ratio test.

(c) $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ By the ratio test, $\lim_{n\to\infty} \frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{3}{n+1} = 0$

11. Find the interval of convergence:

 (a) $\sum_{n=1}^{\infty} \frac{n^2 x^n}{10^n}$ (Ratio) $\lim_{n\to\infty} \frac{(n+1)^2|x|^{n+1}}{10^{n+1}} \cdot \frac{10^n}{n^2|x|^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{|x|}{10} = \frac{|x|}{10}$ The radius of convergence is 10. Check the endpoints:

 If $x = 10$, the sum is $\sum n^2$, which diverges. If $x = -10$, the sum is $\sum(-1)^n n^2$, which still diverges. The interval of convergence is therefore: $(-10, 10)$

 (b) $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}$ (Ratio) $\lim_{n\to\infty} \frac{|3x-2|^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{|3x-2|^n} = \lim_{n\to\infty} \frac{n}{n+1} \frac{|3x-2|}{3} = \frac{|3x-2|}{3}$

 From this, the radius of convergence is 3. The interval so far is: $(-1/3, 5/3)$, so now check the endpoints.

 If $x = -\frac{1}{3}$, the sum becomes: $\sum \frac{(-1)^n}{n}$, which converges (conditionally). If $x = \frac{5}{3}$, the sum becomes the Harmonic Series, which diverges.

 The interval of convergence is $\left[-\frac{1}{3}, \frac{5}{3}\right)$

 (c) $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}$

 Be careful with the indices on this one. If the n^{th} term has index $2n-1$, then the $n+1^{\text{st}}$ index is $2(n+1)-1 = 2n+1$. Furthermore,

$$\frac{(2n-1)!}{(2n+1)!} = \frac{1 \cdot 2 \cdot 3 \cdots (2n-2)(2n-1)}{1 \cdot 2 \cdot 3 \cdots (2n-2)(2n-1)(2n)(2n+1)} = \frac{1}{(2n)(2n+1)}$$

 Now apply the ratio test:

$$\lim_{n\to\infty} \frac{|x|^{2n+1}}{(2n+1)!} \cdot \frac{(2n-1)!}{|x|^{2n-1}} = \lim_{n\to\infty} \frac{x^2}{(2n)(2n+1)} = 0$$

 for all x. Therefore, the interval of convergence is the set of all real numbers.

12. Evaluate the integral. You should be able to do these without the table of integrals.

 (a) $\int \frac{1}{y^2 - 4y - 12}\, dy$

 SOLUTION: Use partial fractions, where

$$\frac{1}{y^2 - 4y - 12} = \frac{1}{(y+2)(y-6)} = \frac{A}{y+2} + \frac{B}{y-6}$$

 Then:

$$1 = A(y-6) + B(y+2) \qquad \Rightarrow \qquad A = -1/8, \quad B = -1/8$$

 Now integrate: $\int \frac{1}{y^2 - 4y - 12}\, dy = -\frac{1}{8} \ln|y+2| + \frac{1}{8} \ln|y-6|$

(b) $\int \frac{2}{3x+1} + \frac{2x+3}{x^2+9}\,dx$
The first integral is a straight u, du substitution with $u = 3x+1$. The second integral is similar to the one on the quiz except that we don't have to complete the square- split the integral into two:

$$\int \frac{2x}{x^2+9}\,dx + \int \frac{3}{x^2+9}\,dx$$

On the first integral, use u, du substitution with $u = x^2+9$, and on the second integral, use either trig substitution (with triangles), or recall the formula for the inverse tangent.
To use trig substitution, we'll use the triangle where $\tan(\theta) = x/3$, so the hypotenuse is $\sqrt{x^2+9}$, $\sec(\theta) = \frac{\sqrt{x^2-9}}{3}$, and $dx = 3\sec^2(\theta)$. The integral becomes:

$$\int \frac{3}{x^2+9}\,dx = \int \frac{9\sec^2(\theta)}{9\sec^2(\theta)}\,d\theta = \theta = \tan^{-1}(x/3)$$

Altogether, we get:

$$\int \frac{2}{3x+1} + \frac{2x+3}{x^2+9}\,dx = \frac{2}{3}\ln|3x+1| + \ln(x^2+9) + \tan^{-1}(x/3) + C$$

(c) $\int x^2\cos(3x)\,dx$ Use integration by parts using a table:

$$\left|\begin{array}{c|c|c} + & x^2 & \cos(3x) \\ - & 2x & \frac{1}{3}\sin(3x) \\ + & 2 & -\frac{1}{9}\cos(3x) \\ - & 0 & -\frac{1}{27}\sin(3x) \end{array}\right| \Rightarrow \int x^2\cos(3x)\,dx = \frac{1}{3}x^2\sin(3x) + \frac{2x}{9}\cos(3x) - \frac{2}{27}\sin(3x) + C$$

(d) $\int_{-2}^{2} |x-1|\,dx$ Here, we want to split the integral at $x = 1$:

$$\int_{-2}^{2} |x-1|\,dx = \int_{-2}^{1} -x+1\,dx + \int_{1}^{2} x-1\,dx = 5$$

(e) $\int \frac{dx}{x\ln(x)}$ Use a u, du substitution with $u = \ln(x)$, $du = \frac{1}{x}\,dx$. This gives:

$$\int \frac{dx}{x\ln(x)} = \int \frac{1}{u}\,du = \ln|u| = \ln(\ln(x)) + C$$

(f) $\int x\sqrt{x-1}\,dx$ It's handy to get rid of square roots where possible. Here, try $u = \sqrt{x-1}$ so that $u^2 = x-1$, or $u^2+1 = x$. This also gives $2u\,du = dx$. Now,

$$\int x\sqrt{x-1}\,dx = \int (u^2+1)\cdot u\cdot 2u\cdot du = \int 2u^4 + 2u^2\,du = \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$$

13. The velocity function is $v(t) = 3t - 5$, $0 \le t \le 3$ (a) Find the displacement. (b) Find the distance traveled.

SOLUTION: The displacement function is the antiderivative of velocity:

$$s(t) = \int 3t - 5\, dt = \frac{3}{2}t^2 - 5t + C$$

where C is the initial displacement. The distance traveled is the integral of the absolute value of velocity:

$$\int_0^3 |3t - 5|\, dt$$

Checking $3t - 5$, we see that it is negative if $t < 5/3$, so we break up the integral:

$$\int_0^3 |3t-5|\, dt = \int_0^{5/3} 5 - 3t\, dt + \int_{5/3}^3 3t - 5\, dt = \frac{25}{6} + \frac{16}{6} = \frac{41}{6}$$

14. Use a template series to find the Maclaurin series for: $\int \frac{e^x - 1}{x}\, dx$

SOLUTION: Starting with the series for e^x, we have:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Therefore, subtract one and divide by x to get:

$$\frac{e^x - 1}{x} = 1 + \frac{1}{2}x + \frac{1}{3!}x^2 + \frac{1}{4!}x^3 + \cdots = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$$

Finally, integrate both sides:

$$\int \frac{e^x - 1}{x}\, dx = C + \sum_{n=1}^{\infty} \frac{x^n}{n! \cdot n}$$

15. Same idea as the previous problem. Start with $1/(1+t^3)$, which looks like the sum for a geometric series (with $r = (-t^3)$), and:

$$\frac{1}{1+t^3} = \sum_{n=0}^{\infty} (-1)^n t^{3n}$$

Now we can multiply both sides by t to get the integrand:

$$\frac{t}{1+t^3} = \sum_{n=0}^{\infty} (-1)^n t^{3n+1}$$

Finally, integrate both sides:

$$\int \frac{t}{1+t^3}\, dt = C + \sum_{n=0}^{\infty} (-1)^n \frac{t^{3n+2}}{3n+2}$$

16. Given y below, find the arc length:

$$y = \int_1^x \sqrt{\sqrt{t} - 1}\, dt \quad \Rightarrow \quad y' = \sqrt{\sqrt{x} - 1} \quad \Rightarrow \quad 1 + (y')^2 = \sqrt{x} - 1 + 1 = \sqrt{x}$$

Therefore,

$$L = \int_1^{16} x^{1/4}\, dx = \frac{4}{5} x^{5/4} \Big|_1^{16} = \frac{4}{5}(2^5 - 1) = \frac{124}{5}$$

17. Note that we're just rotating the curve $y = 1 - x^2$, and not the entire region between the $x-$axis and the curve (like we did for a volume of revolution). Therefore, there is only one surface.

$$r = 2 - y = 2 - (1 - x^2) = 2 + x^2$$

$$ds = \sqrt{1 + (f'(x))^2}\, dx = \sqrt{1 + 4x^2}\, dx$$

The integral is therefore

$$\int_0^1 2\pi(2 + x^2)\sqrt{1 + 4x^2}\, dx$$