Final Exam Review
Calculus II
Sheet 3

1. Determine if the series converges (absolute or conditional) or diverges:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$

   SOLUTION: We might first check to see if it converges absolutely. It will not, by the integral test. To check that the function is decreasing, take the derivative:

$$\frac{\frac{1}{x} \cdot x - 1 \cdot \ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2}$$

   The denominator is always positive for $x > 1$, and the numerator is negative for $x > e$ (which is fine). Now for the integral, perform u, du substitution, and

$$\int_1^{\infty} \frac{\ln(x)}{x}\, dx = \lim_{t\to\infty} \left(\frac{1}{2} (\ln(x))^2 \right|_1^t \to \infty$$

   The integral diverges.

   However, the series is alternating. We've already shown the individual terms $\ln(n)/n$ is decreasing, so now we just need to show they go to zero:

$$\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

   Therefore, the series converges, but only conditionally.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\mathrm{e}^n}{n!}$

   SOLUTION: Use the Ratio Test. Since all terms are positive for $n \geq 1$, we can leave off the absolute value signs:

$$\lim_{n\to\infty} \frac{\mathrm{e}^{n+1}}{(n+1)!} \frac{n!}{\mathrm{e}^n} = \lim_{n\to\infty} \frac{\mathrm{e}}{n+1} = 0 < 1$$

   Therefore, the series converges absolutely.

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{\mathrm{e}^{n^4}}$

   SOLUTION: The exponential in the denominator might be an issue for the ratio test. However, we might see that this is set up for the integral test. If we take the derivative and simplify we get:

$$y(n) = \frac{n^3}{\mathrm{e}^{n^4}} \quad \Rightarrow \quad y'(n) = \frac{n^2}{\mathrm{e}^{n^4}}(3 - 4n^4)$$

   which is negative for $n \geq 1$. We can use the integral test now: $\displaystyle\int_1^{\infty} \frac{x^3}{\mathrm{e}^{x^4}}\, dx$.

Let $u = x^4$ and $du = 4x^3\,dx$. Changing the bounds, if $x = 1$ then $u = 1$, and as $x \to \infty$, so does u:

$$\frac{1}{4}\int_1^{\infty} \mathrm{e}^{-u}\,du = \lim_{t\to\infty} \left(-\frac{1}{4}\mathrm{e}^{-u}\right|_1^t = 0 - -\frac{1}{4} = \frac{1}{4}$$

Therefore, the series converges (absolutely).

(d) $\sum_{n=1}^{\infty} 4^{1-2n}$

SOLUTION: This is a geometric series. Re-writing 4^{1-2n} as ar^n, we get:

$$4^{1-2n} = 4^1 4^{-2n} = 4^1\left(\frac{1}{4^2}\right)^n$$

so this is a geometric series with $a = 4$ and $r = 1/16$. The series then converges absolutely, and it actually converges to:

$$\frac{4\cdot 1/16}{1-(1/16)} = \frac{4}{15}$$

(The question didn't ask for what the sum converges to, but since we knew it, why not provide it?)

2. Let $a_n = \dfrac{n+\ln(n)}{n^2}$.

(a) Does the sequence $\{a_n\}$ converge or diverge? If it converges, find what it converges to.

SOLUTION: We can use l'Hospital's rule to find what the sequence converges to:

$$\lim_{n\to\infty}\frac{n+\ln(n)}{n^2} = \lim_{n\to\infty}\frac{1+1/n}{2n} = \lim_{n\to\infty}\frac{n+1}{2n^2} = \lim_{n\to\infty}\frac{1}{4n} = 0$$

(b) Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge?

SOLUTION: We note that as $n \to \infty$, then $ln(n)/n \to 0$ (by l'Hospital's rule). Therefore, we could show that the series diverges by limit comparison with $1/n$. We have:

$$\lim_{n\to\infty}\frac{n^2+n\ln(n)}{n^2} = \lim_{n\to\infty}\frac{1+\ln/n}{1} = 1+0 = 1$$

Because this limit is between 0 and ∞, our series diverges like the harmonic series.

ALTERNATE SOLUTION: You could use the integral test, and note that

$$\frac{x+\ln(x)}{x^2} = \frac{1}{x} + \frac{\ln(x)}{x^2}$$

3. A bug is crawling along the graph of the curve $y = 3x+1$ for x in the interval $[0,t]$. Find the distance the bug has traveled as a function of t.

SOLUTION: $1 + (y')^2 = 10$, so

$$\int_0^t \sqrt{1+(y')^2}\,dx = \sqrt{10}\,t$$

4. Find the interval of convergence for each of the series:

(a) $\sum_{n=0}^{\infty} \frac{(2x-3)^n}{n\ln(n)}$ (**TYPO: The series should start at** $n=2$...)

SOLUTION: Use the Ratio test, and get

$$\lim_{n\to\infty} \frac{n\ln(n)}{(n+1)\ln(n+1)}|2x-3|$$

Both

$$\lim_{n\to\infty}\frac{n}{n+1}=1 \qquad \lim_{n\to\infty}\frac{\ln(n)}{\ln(n+1)}=\lim_{n\to\infty}\frac{1/n}{1/(n+1)}=\lim_{n\to\infty}\frac{n+1}{n}=1$$

So the ratio converges to $|2x-3|$. Therefore, the series converges absolutely if

$$|2x-3|<1 \quad\Rightarrow\quad \left|2\left(x-\frac{3}{2}\right)\right|<1 \quad\Rightarrow\quad \left|x-\frac{3}{2}\right|<\frac{1}{2}$$

The interval is centered at 3/2 with a radius of 1/2, so the endpoints are $x=1$ and $x=2$, which need to be checked separately:

- At $x=2$, we have:

$$\sum_{n=2}^{\infty}\frac{1}{n\ln(n)} \quad\Rightarrow\quad \int_2^{\infty}\frac{dx}{x\ln(x)}=\int_{\ln(2)}^{\infty}u^{-1}\,du \Rightarrow \lim_{t\to\infty}\left(\ln(u)\right|_{\ln(2)}^{\infty}$$

  which diverges.
- At $x=1$, we have:

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln(n)} \quad\Rightarrow\quad \text{Converges using Alt Series Test}$$

Therefore, the series converges on $[1,2)$

(b) $\sum_{n=0}^{\infty}\frac{x^n}{n+1}$

SOLUTION: The Ratio test gives absolute convergence if $|x|<1$. We check the endpoints manually, at $x=\pm 1$. The interval of convergence is then $[-1,1)$

(c) $\sum_{n=0}^{\infty}\frac{3^n x^n}{5^n}$

SOLUTION: We see that the terms can be written as

$$\left(\frac{3x}{5}\right)^n$$

which makes this a geometric series, and that converges only if

$$\left|\frac{3x}{5}\right|<1 \quad\Rightarrow\quad |x|<\frac{5}{3} \quad\Rightarrow\quad \left(-\frac{5}{3},\frac{5}{3}\right)$$

5. Expand the function $f(x) = \dfrac{2}{4-3x}$ as a power series centered at $x=0$, and determine the values of x for which the series converges.

SOLUTION 1: We could rewrite this to be the sum of a geometric series:

$$\frac{2}{4}\frac{1}{1-\left(\frac{3x}{4}\right)} = \frac{1}{2}\frac{1}{1-(3x/4)} \qquad \Rightarrow \qquad \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{3x}{4}\right)^n$$

And this converges only for $|3x/4| < 1$, or $|x| < 4/3$.

SOLUTION 2: We could compute out the terms of the Maclaurin series:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$2(4-3x)^{-1}$	$1/2$
1	$(-2)(-3)(4-3x)^{-2}$	$(1/2)(3/4)$
2	$(-2)(-3)^2(-2)(4-3x)^{-3}$	$(1/2)(3/4)^2(2)$
3	$(-2)(-3)^3(-2)(-3)(4-3x)^{-4}$	$(1/2)(3/4)^3(2\cdot 3)$

$$\Rightarrow \qquad f^{(n)}(0) = \frac{1}{2}\cdot\left(\frac{3}{4}\right)^n\cdot n!$$

From which the series becomes:

$$\sum_{n=0}^{\infty}\frac{1}{2}\cdot\left(\frac{3}{4}\right)^n n!\cdot\frac{1}{n!}x^n = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{3x}{4}\right)^n$$

6. Evaluate the integral:

(a) $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

SOLUTION: With an x^2 in the numerator, it might be easiest to go with a trig substitution (if only x in the numerator, we might use a regular substitution). In this case, let $x = 4\sin(\theta)$ so that

$$\sqrt{16-x^2} = 16 - 16\sin^2(\theta) = 16\cos^2(\theta)$$

Furthermore, $dx = 4\cos(\theta)\,d\theta$, so we get:

$$\int \frac{16\sin^2(\theta)\,4\cos(\theta)d\theta}{4\cos(\theta)} = 16\int\sin^2(\theta)d\theta = 8\int 1-\cos(2\theta)\,d\theta = 8\theta - 4\sin(2\theta)$$

$$= 8\theta - 8\sin(\theta)\cos(\theta) = 8\sin^{-1}\left(\frac{x}{4}\right) - 8\frac{x}{4}\frac{\sqrt{16-x^2}}{4} = 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{1}{2}x\sqrt{16-x^2} + C$$

(b) $\displaystyle\int \sin^2(x)\cos^3(x)\,dx$

SOLUTION: Keep one $\cos(x)$ reserved for a u, du substitution, with $u = \sin(x)$:

$$\int \sin^2(x)\cos^2(x)\cos(x)\,dx = \int\sin^2(x)(1-\sin^2(x)\ \cos(x)\,dx = \int u^2(1-u^2)\,du$$

$$= \frac{1}{3}u^3 - \frac{1}{5}u^5 = \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C$$

(c) $\int x^2 e^{-2x}\, dx$

Integrate by parts (using a table) to get:

$$-\frac{1}{4}e^{-2x}(1 + 2x + 2x^2) + C$$

(d) $\int \tan^{-1}(x)\, dx$

Integrate by parts (once) to get:

$$x\,\tan^{-1}(x) - \frac{1}{2}\ln(1 + x^2) + C$$

(e) $\int \frac{x^2 - x + 1}{x^2 + x}\, dx$

SOLUTION: Do long division first, and we get:

$$\int 1 + \frac{1 - 2x}{x^2 + x}\, dx = \int 1 + \frac{1 - 2x}{x(x + 1)}\, dx$$

The second integral is done by partial fractions:

$$\frac{1 - 2x}{x(x + 1)} = \frac{A}{x} + \frac{B}{x + 1} \qquad \Rightarrow \qquad 1 - 2x = A(x + 1) + Bx$$

From which we get $A = 1$, $B = -3$. The full integral is then:

$$x + \ln|x| - 3\ln|x + 1| + C$$

(f) $\int \frac{dx}{x^2 + 4x - 5}$

SOLUTION: You could use a trig substitution (and complete the square), but the denominator factors, so we can use partial fractions:

$$\frac{1}{x^2 + 4x - 5} = \frac{1}{(x + 5)(x - 1)} = \frac{A}{x + 5} + \frac{B}{x - 1} \qquad \Rightarrow \qquad 1 = A(x - 1) + B(x + 5)$$

From which we get $A = -1/6$, $B = 1/6$ and the integral is:

$$-\frac{1}{6}\ln|x + 5| + \frac{1}{6}\ln|x - 1| + C$$

(g) $\int_0^3 |x^2 - 4|\, dx$

SOLUTION: Break it up at $x = 2$:

$$\int_0^2 4 - x^2\, dx + \int_2^3 x^2 - 4\, dx = \cdots = \frac{23}{3}$$

(h) $\int_1^9 \frac{\sqrt{x} - 2x^2}{x}\, dx$

SOLUTION: The denominator is x^{-1}, which can be multiplied through:

$$\int_1^9 (x^{1/2} - 2x^2)x^{-1}\, dx = \int_1^9 x^{-1/2} - 2x\, dx = \cdots = -76$$

(i) $\displaystyle\int_{-3}^{3} \frac{\sin(x)}{x^2+1}\,dx$

SOLUTION: Kind of a trick question- The integrand is odd, so the integral is 0.

7. Evaluate $\displaystyle\int \frac{dx}{x^2-1}\,dx$ two ways- Using partial fractions and using trig substitution.

SOLUTION: Using partial fractions, we get

$$-\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$

Using trig substitution, with $x = \sec(\theta)$, we get:

$$\int \frac{\sec(\theta)\tan(\theta)\,d\theta}{\tan^2(\theta)} = \int \csc(\theta)\,d\theta = \ln|\csc(\theta) - \cot(\theta)| + C$$

Back substitute for x:

$$\ln\left|\frac{x}{\sqrt{x^2-1}} - \frac{1}{\sqrt{x^2-1}}\right| = \ln\left|\frac{x-1}{\sqrt{x^2-1}}\right| = \ln\left|\frac{\sqrt{x-1}}{\sqrt{x+1}}\right| + C$$

8. Determine if the integral converges or diverges. If it converges, determine what it converges to. $\displaystyle\int_{-\infty}^{9} \mathrm{e}^{4x}\,dx$

SOLUTION: Converges to $\frac{1}{4}\mathrm{e}^{36}$

9. Find a series for $x\tan^{-1}(x^2)$ using $1/(1+x^2)$.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

Therefore,

$$\tan^{-1}(x) = \int \frac{1}{1+x^2}\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

Similarly, solving for C (put in $x=0$), we see that $C=0$.

Now, substitute x^2 for x

$$\tan^{-1}(x^2) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{2n+1}$$

so finally we have:

$$x\tan^{-1}(x^2) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{2n+1}$$

10. Consider the region in the first quadrant bounded by the curve $y = 9 - x^2$ with $0 \le x \le 3$. Consider the solid obtained by rotating that region about the x axis. Set up two integrals that represent the volume of this solid- One using shells, and one using disks.

For disks, we'll be integrating in x, and for shells, we'll be integrating in y:

$$\text{Shells: } \int_0^9 2\pi y\sqrt{9-y}\,dy$$

$$\text{Disks: } \int_0^3 \pi(9-x^2)^2\,dx$$

11. Same region as before. Set up an integral representing the volume (using any appropriate technique) if the region is revolving about $x = 4$, and then if the region is revolving about $y = -2$.

   - For $x = 4$, you could use shells or washers. With shells, the solution is:

   $$\int 2\pi(4 - x)(9 - x^2)\, dx$$

   - For $y = -2$, if you want to stay with x, you can use washers, whose inner radius is 2, and outer radius is $y + 2 = 11 - x^2$:

   $$\int_0^3 \pi((11 - x^2)^2 - 2^2)\, dx$$

12. Use differentiation to find a power series for $f(x) = 1/(1 + x)^2$:

   SOLUTION: We notice that $1/(1 + x)^2$ is just about the derivative of $1/(1 + x)$:

   $$\frac{d}{dx}\frac{1}{1 + x} = \frac{-1}{(1 + x)^2}$$

   Therefore, if we have a series for $1/(1 + x)$, we differentiate it and multiply by -1 to get the desired series. We start then with $1/(1 + x)$.

   $$\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

   Differentiate both sides:

   $$\frac{-1}{(1 + x)^2} = \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$$

   Finally, multiply by -1:

   $$\frac{1}{(1 + x)^2} = \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1}$$

13. Use the *definition* of the definite integral (with right endpoints) to calculate the value of $\int_0^2 (x^2 - x)\, dx$.

   (Hint: The formulas for $\sum i^2$ and $\sum i^3$ would be given to you).

   SOLUTION: The i^{th} right endpoint is $0 + i\frac{2}{n}$, or $2i/n$. Putting this into the Riemann sum and taking the limit, we have

   $$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i^2}{n^2} - \frac{2i}{n}\right)\frac{2}{n} = \lim_{n\to\infty} \left[\frac{8}{n^3}\sum_{i=1}^{n} i^2 - \frac{4}{n^2}\sum_{i=1}^{n} i\right]$$

   $$= \lim_{n\to\infty} \left[\frac{8}{n^3}\frac{n(n + 1)(2n + 1)}{6} - \frac{4}{n^2}\frac{n(n + 1)}{2}\right] = \frac{8}{3} - 2 = \frac{2}{3}$$

14. Find the derivative of the function : $y = \int_{\sqrt{x}}^{x} \frac{e^t}{t}\, dt$

SOLUTION: Use the formula we derived, or derive it using the technique from pg 390:

$$\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{\sqrt{x}} \frac{1}{2\sqrt{x}} - \frac{e^x}{x} = \frac{e^{\sqrt{x}}}{2x} - \frac{e^x}{x}$$

15. Find the c guaranteed by the Mean Value Theorem for Integrals, if $f(x) = 1/x$ on the interval $[1, 3]$. Hint: It has something to do with the average value of f.

SOLUTION: The Mean Value Theorem for Integrals says that if f is continuous on $[a, b]$ then there is a c in $[a, b]$ so that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\, dx \quad \Rightarrow \quad \frac{1}{c} = \frac{1}{3-1}\int_1^3 \frac{1}{x}\, dx \Rightarrow \frac{1}{c} = \frac{1}{2}\ln(x)|_1^3 = \frac{1}{2}(\ln(3) - \ln(1))$$

Therefore, $c = \frac{2}{\ln(3)}$ (which is about 1.8).

16. What is wrong with the following proof:

Proof by induction that $n + 1 < n$:

Assume true for $n = k$, so that $k + 1 < k$. We show that this implies $k + 2 < k + 1$:

Since $k+2 = k+1+1 = (k+1)+1 < k+1$ by induction, then $k+1 < k$ for all positive integers k.

SOLUTION: We didn't prove the "base" case- And in this instance, there is no base case (since $n + 1$ is smaller than n!).