Exam 2 Review

The exam will cover sections 4.9 (antiderivatives), 5.1-5.5, 6.1-6.5. The following questions are not meant to be exhaustive, so you should also be sure you've looked over your old quizzes and understand the homework.

Recall that there are several methods for evaluating the integral that are handy to keep in mind. They relate to the two ways of looking at integration, and lastly, we looked at some "applications" of the integral.

1. The integral as area (this is the definition of the integral).
   - Integrate using geometry (area of circles or triangles, etc.)
   - Integrate using symmetry (even, odd) of the integrand.
   - Integrate using the definition (the Riemann Sum) directly.
2. The integral as antiderivative (the indefinite integral, the FTC)
   - Integrate using the table (that we've memorized)
   - Simplify first, then integrate.
   - u, du substitution.
3. Applications of the integral:
   - Area between curves
   - Volumes of objects with known cross-sections (like squares or triangles)
   - Volumes of solids of revolution. Use disks, washers and/or shells.
   - The average value of f
   - Work.

For the sake of time, often on exam questions referring to area, I will ask you to *set up, but do not evaluate* the integral(s) you would need to compute to find the area of a given region (graphs of things more complicated than a parabola or line would be provided). Be sure you're reading the question carefully so that you don't spend a lot of time doing unnecessary computations.

I will provide the sum formulas for $\sum i^2$ and $\sum i^3$ if you need them. You should memorize the table of integrals from 5.4 with the exception of the hyperbolic trig functions ($\sinh(x)$ and $\cosh(x)$), which we did not cover.

Here are your review questions. The first two questions are easiest if we have the theorem and definition memorized.

Review Questions

1. State the Fundamental Theorem of Calculus:
2. Give the *definition* of the definite integral $\int_a^b f(x)\,dx =$
3. Find the area bounded between the regions $y = 1 - 2x^2$ and $y = |x|$.
4. For each of the following integrals, write the definition using the Riemann sum, and then evaluate them (MUST use the limit of the Riemann sum for credit, and do not re-write them using the properties of the integral). Note that:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

   (a) $\int_2^5 x^2\,dx$

   (b) $\int_1^3 1 - 3x\,dx$

   (c) $\int_0^5 1 + 2x^3\,dx$

5. Evaluate the integral and interpret it as the area of a region (sketch it).

$$\int_0^4 |\sqrt{x+2} - x|\,dx$$

6. True or False (and give a short reason):

   (a) $\int_0^2 (x - x^3)\,dx$ represents the area under the curve $y = x - x^3$ from 0 to 2.

   (b) If $3 \leq f(x) \leq 5$ for all x, then $6 \leq \int_1^3 f(x)\,dx \leq 10$

   (c) If f, g are continuous on $[a, b]$, then

$$\int_a^b f(x) - g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

   (d) The fact that f, g were each individually continuous on $[a, b]$ was an important thing to state in the last problem.

   (e) If f, g are continuous on $[a, b]$, then

$$\int_a^b f(x)g(x)\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$

   (f) All continuous functions have derivatives.

   (g) All continuous functions have antiderivatives.

7. For each of the following Riemann sums, evaluate the limit by first recognizing it as an appropriate integral:

   (a) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3}{n}\right) \sqrt{1+\frac{3i}{n}}$ (Find four different integrals for this one!)

   (b) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(2+3\cdot\frac{25i^2}{n^2}\right)\left(\frac{5}{n}\right)$

8. Evaluate the integral, if it exists

   (a) $\int_1^9 \frac{\sqrt{u}-2u^2}{u}\,du$

   (b) $\int 3^x + \frac{1}{x} + \sec^2(x)\,dx$

   (c) $\int_{-\pi/4}^{\pi/4} \frac{t^4\tan(t)}{2+\cos(t)}\,dt$

   (d) $\int_0^3 |x^2-4|\,dx$

   (e) $\int \frac{\cos(\ln(x))}{x}\,dx$

   (f) $\int_0^2 \sqrt{4-x^2}\,dx$

   (g) $\int \frac{1}{\sqrt{1-x^2}}\,dx$

   (h) $\int_{-1}^2 \frac{1}{x}\,dx$

   (i) $\int_0^1 (\sqrt[4]{w}+1)^2\,dw$

   (j) $\int_{-2}^{-1} \frac{1}{x}\,dx$

   (k) $\int_0^{1/2} \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dt$

   (l) $\int (1+\tan(t))\sec^2(t)\,dt$

   (m) $\int \tan(x)\,dx$

   (n) $\int x\sqrt{1+x}\,dx$

   (o) $\int \frac{y-1}{\sqrt{3y^2-6y+4}}\,dy$

   (p) $\int_{-1}^4 |t-3|\,dt$

9. Find the derivative of the function:

   (a) $F(x) = \int_0^{x^2} \frac{\sqrt{t}}{1+t^2}\,dt$

   (b) $y = \int_{\sqrt{x}}^{3x} \frac{e^t}{t}\,dt$

10. Evaluate by recognizing this as a Riemann Sum:

$$\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \cdots \left(\frac{n}{n}\right)^9\right]$$

11. Evaluate:

   (a) $\int_0^1 \frac{d}{dx}\left(e^{\tan^{-1}(x)}\right)\,dx$

   (b) $\frac{d}{dx}\int_0^1 e^{\tan^{-1}(x)}\,dx$

   (c) $\frac{d}{dx}\int_0^x e^{\tan^{-1}(t)}\,dt$

12. (a) Sketch the graph of $f(x) = |x| - 1$.
    (b) Suppose this function is the derivative of some other function, $F(x)$. Sketch one possibility using your previous graph as a guide.
    (c) Sketch the function $G(x) = \int_{-2}^{x} f(t)\,dt$ for the same values of $-2 \leq x \leq 4$, again using your previous answers as a guide.
    (d) What is the relationship (if any) between F and G?

13. A particle moves along a line with velocity $v(t) = t^2 - t$, where v is measured in meters per second. Find (a) the displacement and (b) the distance traveled by the particle during the time interval $[0, 5]$.

14. If $f(x) = x^2$, find the average value of f on the interval $[1, 3]$. Find the value(s) of c guaranteed by the Mean Value Theorem for Integrals.

15. If f is continuous and $\displaystyle\int_0^9 f(x)\,dx = 4$, find $\displaystyle\int_0^3 xf(x^2)\,dx$

16. If $f''(x) = 2 - 12x$, $f(0) = 0$ and $f(2) = 15$, find $f(x)$.

17. A cross section of a tank of water is the bottom half of a circle of radius 10 ft, and is 50 ft long. Find the work done in pumping the water over the rim of the tank if it filled to a depth of 7 feet (set up the integral only, water weighs 62.5 lbs per cubic feet.) Set up the integral if we were pumping the water up an additional 10 feet up.

18. A 10 ft chain weighs 25 lbs and hangs from a ceiling. Find the work done in lifting the lower end of the chain to the ceiling to that it's level with the upper end.

19. A heavy rope, 20 meters long, weighs 0.5 kg/m and hangs over a building that is 40 meters tall. (Recall that $g = 9.8\text{m/s}^2$)
    (a) How much work is done pulling the rope to the top?
    (b) How much work is done pulling half of the rope to the top? (Hint: It makes sense that it is not half your previous answer, right?)

20. Let R be the region in the first quadrant bounded by $y = x^3$ and $y = 2x - x^2$. Calculate the following quantities: (Exam note: Region R would typically be plotted for you).
    (a) The area of R.
    (b) Volume obtained by rotating R about the $x-$axis.
    (c) Volume obtained by rotating R about the $y-$axis.

21. Use any method to find an integral representing the volume generated by rotating the given region about the specified axis. You do NOT need to evaluate the integral:
    (a) $y = \sqrt{x}, y = 0, x = 1$; about $x = 2$.
    (b) $y = x^2$, $y = 2 - x^2$; about $x = 1$.
    (c) $y = x^2$, $y = 2 - x^2$; about $y = -3$.
    (d) $y = \tan(x)$, $y = x$, $x = \pi/3$; about the $y-$axis.