Quiz 3 (Section A) Solutions

1. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

   SOLUTION: Use the Ratio Test.

$$\lim_{n\to\infty} \frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2} = \lim_{n\to\infty} \frac{3(n+1)^2}{(n+1)n^2} = \lim_{n\to\infty} \frac{3(n+1)}{n^2} = \lim_{n\to\infty} \frac{3}{2n} = 0$$

   (The last step was using l'Hospital's rule)

   Therefore, the series converges (absolutely) by the Ratio Test.

2. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$

   SOLUTION: As $n \to \infty$, the terms a_n go to $\pm 3/2$ (so the limit does not exist). Therefore, the series diverges by the Test for Divergence.

3. $\displaystyle\sum_{k=1}^{\infty} k^2 \mathrm{e}^{-k}$

   SOLUTION: Use the Ratio Test.

$$\lim_{k\to\infty} \frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^2 \cdot \frac{1}{e} = \frac{1}{e}$$

   The limit is less than 1, so the series converges (absolutely) by the Ratio Test.

4. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

   SOLUTION: First, we note that the series does not converge absolutely. To see this, we can use the comparison test with $\sum 1/\sqrt{n}$, which is a divergent $p-$series.

$$\sqrt{n} - 1 < \sqrt{n} \quad \Rightarrow \quad \frac{1}{\sqrt{n-1}} > \frac{1}{\sqrt{n}}$$

   Therefore, by the direct comparison test, the series does not converge absolutely.

   Now we can apply the Alternating Series Test for conditional convergence. Here, our $b_n = 1/(\sqrt{n} - 1)$.

   - b_n is decreasing: Since $\sqrt{n+1} - 1 > \sqrt{n} - 1$, then

$$\frac{1}{\sqrt{n+1}-1} < \frac{1}{\sqrt{n}}$$

   - The limit of b_n is zero:

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}-1} = 0$$

   Therefore, by the Alternating Series Test, the series converges conditionally.