Solutions to in-class problems for 7.3

1. $\displaystyle\int_0^3 \sqrt{9-x^2}\,dx$

   SUBSTITUTION: $x = 3\sin(\theta)$, and $dx = 3\cos(\theta)d\theta$.

   Also, if $x = 0$, then $\sin(\theta) = 0$, or $\theta = 0$. Similarly, if $x = 3$, then $\sin(\theta) = 1$, which occurs when $\theta = \pi/2$.

   $$\int_0^{\pi/2} 3\cos(\theta)\cdot 3\cos(\theta)\,d\theta = 9\int \cos^2(\theta)\,d\theta$$

   For practice in 7.2, we use the half angle formla: $\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta)$:

   $$9\int_0^{\pi/2} \frac{1}{2}(1+\cos(2\theta))\,d\theta = \left(\frac{9}{2}\theta + \frac{9}{4}\sin(2\theta)\right|_0^{\pi/2} = \frac{9}{4}\pi$$

   We note that because this was a definite integral, we did not need to convert back to x.

2. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-9}}$

   Make the substitution:

   $$x = 3\sec(\theta) \qquad dx = 3\sec(\theta)\tan(\theta)\,d\theta \qquad \sqrt{x^2-9} = 3\tan(\theta)$$

   Therefore, we have:

   $$\int \frac{3\sec(\theta)\tan(\theta)\,d\theta}{9\sec^2(\theta)\cdot 3\tan(\theta)} = \frac{1}{9}\int \cos(\theta)\,d\theta = \frac{1}{9}\sin(\theta) + C$$

   Convert back using the reference triangle:

   (opposite is $\sqrt{x^2-9}$, adjacent is 3, hypotenuse is x.)

   $$\sin(\theta) = \frac{\sqrt{x^2-9}}{x}$$

   Therefore,

   $$\int \frac{dx}{x^2\sqrt{x^2-9}} = \frac{1}{9}\sin(\theta) + C = \frac{\sqrt{x^2-9}}{9x} + C$$

3. $\displaystyle\int \sqrt{4x^2+20}\,dx$

   Let $2x = \sqrt{20}\tan(\theta)$ so $dx = \frac{\sqrt{20}}{2}\sec^2(\theta)\,d\theta$, and the integrand becomes:

   $$\sqrt{4x^2+20} = \sqrt{20(\tan^2(\theta)+1)} = \sqrt{20}\sec(\theta)$$

and the integral becomes

$$\sqrt{20}\frac{\sqrt{20}}{2}\int \sec^3(\theta)\,d\theta = 10\int \sec^3(\theta)d\theta$$

To integrate this, use the table from the handout in class:

$$\int \sec^n(u)\,du = \frac{1}{n-1}\sec^{n-2}(u)\tan(u) + \frac{n-2}{n-1}\int \sec^{n-2}(u)\,du$$

$$\int \sec^3(\theta)\,d\theta = \frac{1}{2}\sec(\theta)\tan(\theta)+\frac{1}{2}\int \sec(\theta)\,d\theta = \frac{1}{2}\sec(\theta)\tan(\theta)+\frac{1}{2}\ln|\sec(\theta)+\tan(\theta)|$$

to find that we have:

$$\int \sqrt{4x^2+20}\,dx = 10\int \sec^3(\theta)\,d\theta = 5\tan(\theta)\sec(\theta)+5\ln|\sec(\theta)+\tan(\theta)|+C$$

For the reference triangle, the side opposite θ is x, adjacent is $\sqrt{5}$, hypotenuse is $\sqrt{x^2+5}$. Therefore,

$$\tan(\theta) = \frac{opp}{adj} = \frac{x}{\sqrt{5}} \qquad\qquad \sec(\theta) = \frac{hyp}{adj} = \frac{\sqrt{x^2+5}}{\sqrt{5}}$$

Substituting these back, we have:

$$\int \sqrt{4x^2+20}\,dx = 5\cdot\frac{x}{\sqrt{5}}\cdot\frac{\sqrt{x^2+5}}{\sqrt{5}} + 5\ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}}+\frac{x}{\sqrt{5}}\right| =$$

$$x\sqrt{x^2+5}+5\ln|\sqrt{x^2+5}+x| - 5\ln(\sqrt{5})+C$$

4. $\displaystyle\int \frac{dx}{((x-3)^2+2)^2}$

The substitutions are:

$$x-3 = \sqrt{2}\tan(\theta) \qquad dx = \sqrt{2}\sec^2(\theta)\,d\theta \qquad (x-3)^2+2 = 2(\tan^2(\theta)+1)$$

To get

$$\int \frac{\sqrt{2}\sec^2(\theta)\,d\theta}{4\sec^4(\theta)} = \frac{1}{2\sqrt{2}}\int \cos^2(\theta)\,d\theta = \frac{1}{4\sqrt{2}}\int (1+\cos(2\theta))\,d\theta =$$

$$\frac{1}{4\sqrt{2}}\left(\theta+\frac{1}{2}\sin(2\theta)\right)$$

Now, $\theta = \tan^{-1}\left(\frac{x-3}{\sqrt{2}}\right)$ and using the fact that $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$, on the triangle we get

$$\sin(\theta) = \frac{x-3}{\sqrt{x^2-6x+11}} \qquad \cos(\theta) = \frac{\sqrt{2}}{\sqrt{x^2-6x+11}}$$

Overall, the solution is:

$$\frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{x-3}{\sqrt{2}}\right) + \frac{1}{4}\,\frac{(x-3)}{x^2-6x+11}+C$$

5. $\displaystyle\int \frac{t^5}{\sqrt{t^2+2}}\, dt$

SOLUTION: Think of "$\tan^2(\theta)+1$", and let $t = \sqrt{2}\tan(\theta)$ and $dt = \sqrt{2}\sec^2(\theta)\, d\theta$. Putting these expressions in, we get:

$$\int \frac{2^{5/2}\tan^5(\theta)}{\sqrt{2\tan^2(\theta)+2}}\sqrt{2}\sec^2(\theta)\, d\theta = \frac{2^3}{\sqrt{2}}\int \tan^5(\theta)\sec(\theta)\, d\theta$$

To continue, reserve $\sec(\theta)\tan(\theta)$ for a substitution: $u = \sec(\theta)$, and

$$\tan^4(\theta) = (\sec^2(\theta)-1)^2$$

so that the integral becomes:

$$2^{5/2}\int (u^2-1)^2\, du = 2^{5/2}\int u^4 - 2u^2 + 1\, du = 2^{5/2}\left(\frac{1}{5}u^5 - \frac{2}{3}u^3 + u\right) + C$$

where $u = \sec(\theta)$, and after that, $\tan(\theta) = \frac{t}{\sqrt{2}}$, so that $\sec(\theta) = \frac{\sqrt{t^2+2}}{\sqrt{2}}$.

ALTERNATE SOLUTION:

We actually could have set $w = t^2+2$ and done everything by substitution.

6. $\displaystyle\int_0^1 \sqrt{1+x^2}\, dx$

Let $x = \tan(\theta)$, so that $dx = \sec^2(\theta)\, d\theta$ and $1+x^2 = \sec^2(\theta)$. Note that we also have bounds:

$$\tan(\theta) = 0 \quad \Rightarrow \quad \theta = 0$$

$$\tan(\theta) = 1 \quad \Rightarrow \quad \theta = \frac{\pi}{4}$$

Putting it all together:

$$\int_0^{\pi/4} \sec^3(\theta)\, d\theta$$

(We would use the table or integration by parts with $dv = \sec^2(\theta)$ (See bottom of p. 475)).

7. $\displaystyle\int \frac{dx}{x^4\sqrt{x^2-2}}$

SOLUTION: Think of $\sec^2(\theta) - 1 = \tan^2(\theta)$. We will substitute $x = \sqrt{2}\,\sec(\theta)$ and $dx = \sqrt{2}\sec(\theta)\tan(\theta)\, d\theta$.

$$\int \frac{\sqrt{2}\sec(\theta)\tan(\theta)\, d\theta}{4\sec^4(\theta)\sqrt{2}\tan(\theta)} = \frac{1}{4}\int \cos^3(\theta)\, d\theta$$

To integrate, rewrite the integrand as $(1-\sin^2(\theta))\cos(\theta)$