Extra Practice with Limits

1. Compute each limit (algebraically) if it exists:

   (a) $\lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$

   (b) $\lim_{x\to 2} \frac{x^2-4}{\frac{1}{x}-\frac{1}{2}}$

   (c) $\lim_{x\to 2} \frac{x^2-2x-3}{x-3}$

   (d) $\lim_{x\to\infty} (\sqrt{x^2+2x} - x)$

   (e) $\lim_{x\to\infty} \frac{3x+2}{\sqrt{x^2-1}}$

2. Determine if the sequence converges, and if it does, find the limit:

   (a) $a_n = \mathrm{e}^{1/n}$

   (b) $a_n = \frac{2^{n+3}}{3^n}$

   (c) $a_n = \frac{2^n}{n!}$

   (d) $\{1, 0, -1, 0, 1, 0, -1, \cdots\}$

   (e) $a_n = \frac{(n+1)(2n+1)}{n^2+2n+1}$

3. Find the limit:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(3\left(\frac{2i}{n}\right)+1\right)$$

   Hint: Expand the sum, then use the sum formulas to get an expression involving n only (with no sums).

SOLUTIONS:

1. Compute each limit (algebraically) if it exists:

   (a) $\lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}$ Simplify the expression first (common denominator):

$$\lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}=\lim_{h\to 0}\frac{1}{h}\cdot\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\cdot\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}=$$

$$\lim_{h\to 0}\frac{1}{h}\cdot\frac{x-(x+h)}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}=\frac{-1}{x\cdot 2\sqrt{x}}=\frac{-1}{2x^{3/2}}$$

   (b) $\lim_{x\to 2}\frac{x^2-4}{\frac{1}{x}-\frac{1}{2}}$ Simplify the denominator then simplify before taking the limit:

$$\lim_{x\to 2}\frac{x^2-4}{\frac{1}{x}-\frac{1}{2}}=\lim_{x\to 2}\frac{x^2-4}{\frac{2-x}{2x}}=\lim_{x\to 2}\frac{(x+2)(x-2)2x}{2-x}=\lim_{x\to 2}-(x+2)2x=-16$$

   (c) $\lim_{x\to 2}\frac{x^2-2x-3}{x-3}$

$$\lim_{x\to 2}\frac{x^2-2x-3}{x-3}=\frac{4-4-3}{2-3}=3$$

   (Be sure to check the numbers before doing any algebra!)

   (d) $\lim_{x\to\infty}(\sqrt{x^2+2x}-x)$ Multiply by the conjugate:

$$\lim_{x\to\infty}\sqrt{x^2+2x}-x\cdot\frac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x}=\lim_{x\to\infty}\frac{(x^2+2x)-x^2}{\sqrt{x^2+2x}+x}=\lim_{x\to\infty}\frac{2x}{\sqrt{x^2+2x}+x}=$$

$$\lim_{x\to\infty}\frac{2x/x}{(\sqrt{x^2+2x}+x)/x}=\lim_{x\to\infty}\frac{2}{\sqrt{\frac{x^2+2x}{x^2}}+1}=\lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{2}{x}}+1}=\frac{2}{2}=1$$

   (e) $\lim_{x\to\infty}\frac{3x+2}{\sqrt{x^2-1}}$ Divide numerator and denominator by x:

$$\lim_{x\to\infty}\frac{3x+2}{\sqrt{x^2-1}}=\lim_{x\to\infty}\frac{3+\frac{2}{x}}{(\sqrt{x^2-1})/x}=\lim_{x\to\infty}\frac{3+\frac{2}{x}}{\sqrt{\frac{x^2-1}{x^2}}}=\lim_{x\to\infty}\frac{3+\frac{2}{x}}{\sqrt{1-\frac{1}{x^2}}}=3$$

2. Determine if the sequence converges, and if it does, find the limit:

   (a) $a_n=\mathrm{e}^{1/n}$
   SOLUTION: The exponential function is continuous at zero, so:

$$\lim_{n\to\infty}\mathrm{e}^{1/n}=\mathrm{e}^{\lim_{n\to\infty}(1/n)}=\mathrm{e}^0=1$$

(b) $a_n = \dfrac{2^{n+3}}{3^n}$

SOLUTION: $0 < 2/3 < 1$, so $(2/3)^n \to 0$ as $n \to \infty$. Therefore:

$$\lim_{n\to\infty} \frac{2^{n+3}}{3^n} = \lim_{n\to\infty} \frac{2^n 2^3}{3^n} = 8 \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$$

(c) $a_n = \dfrac{2^n}{n!}$

SOLUTION: If we examine the sequence term by term:

$$\begin{aligned} a_1 &= \tfrac{2}{1} \\ a_2 &= \tfrac{2\cdot 2}{1\cdot 2} \\ a_3 &= \tfrac{2\cdot 2\cdot 2}{1\cdot 2\cdot 3} \\ a_4 &= \tfrac{2\cdot 2\cdot 2\cdot 2}{1\cdot 2\cdot 3\cdot 4} \end{aligned} \qquad \Rightarrow \qquad a_{n+1} = \frac{2}{n+1} a_n$$

And we see that $a_n \leq 2$ for all n. We could use a kind of squeeze theorem:

$$0 \leq a_{n+1} = \frac{2}{n+1} a_n \leq \frac{2}{n+1} \cdot 2 \to 0$$

(d) $\{1, 0, -1, 0, 1, 0, -1, \cdots\}$

SOLUTION: The limit does not exist.

(e) $a_n = \dfrac{(n+1)(2n+1)}{n^2+2n+1}$

SOLUTION: Use either l'Hospital's rule or divide numerator and denominator by n^2 to get the limit is 2.

3. Find the limit:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \left(3\left(\frac{2i}{n}\right) + 1\right)$$

Hint: Expand the sum, then use the sum formulas to get an expression involving n only (with no sums).

SOLUTION: Using the hint, we'll do the algebra first:

$$\sum_{i=1}^{n} \frac{2}{n} \left(3\left(\frac{2i}{n}\right) + 1\right) = \sum_{i=1}^{n} \frac{12i}{n^2} + \frac{2}{n} = \frac{12}{n^2} \sum_{i=1}^{n} i + \frac{2}{n} \sum_{i=1}^{n} 1 = \frac{12}{n^2} \frac{n(n+1)}{2} + \frac{2}{n} \cdot n = \frac{6(n+1)}{n} + 2$$

Finally, taking the limit as $n \to \infty$, we get $6 + 2 = 8$.