Exam 2 Review Solutions

1. State the Fundamental Theorem of Calculus: Let f be continuous on $[a,b]$.

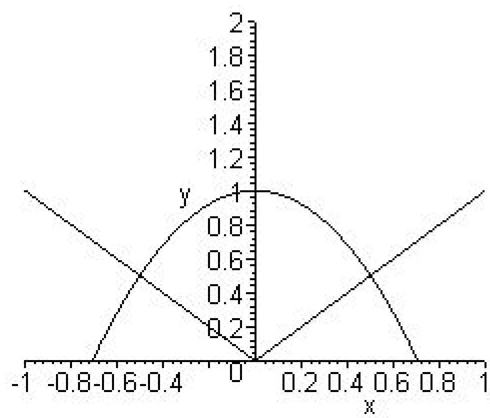
   - If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
   - $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f.

2. Give the *definition* of the definite integral: $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x_i$ Or you may be more specific and use right endpoints:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f\left(a + \frac{b-a}{n} i\right)\left(\frac{b-a}{n}\right)$$

3. Find the area bounded between the regions $y = 1 - 2x^2$ and $y = |x|$.

   The curve intersects with the line $y = x$ where $1 - 2x^2 = x$, or $2x^2 + x - 1 = 0$, so that $x = \frac{1}{2}$ or $x = -1$ (you can use the quadratic formula if you would like). We choose $x = \frac{1}{2}$. By symmetry, we only need to integrate over the right half of the region shown in the Figure to the right, then multiply by 2:



$$2\int_0^{1/2} (1-2x^2) - x\,dx = 2\left. x - \frac{2}{3}x^3 - \frac{1}{2}x^2 \right|_0^{1/2}$$

$$= 2\left(\frac{1}{2} - \frac{2}{3}\cdot\frac{1}{8} - \frac{1}{2}\cdot\frac{1}{4}\right) = 2\frac{7}{24} = \frac{7}{12}$$

4. For each of the following integrals, write the definition using the Riemann sum, and then evaluate them (MUST use the limit of the Riemann sum for credit, and do not re-write them using the properties of the integral):

   (a) $\int_2^5 x^2\,dx$

$$\lim_{n\to\infty} \sum_{i=1}^n \left(2 + \frac{3i}{n}\right)^2 \frac{3}{n} = \lim_{n\to\infty} \sum_{i=1}^n \left(4 + \frac{12i}{n} + \frac{9i^2}{n^2}\right)\frac{3}{n} =$$

$$\lim_{n\to\infty} \sum_{i=1}^n \left(12 + 18\cdot\frac{n+1}{n} + \frac{27}{6}\cdot\frac{(n+1)(2n+1)}{n^2}\right) = 39$$

   (b) $\int_1^3 1 - 3x\,dx$

$$\lim_{n\to\infty} \sum_{i=1}^n \left[1 - 3\left(1 + \frac{2i}{n}\right)\right]\frac{2}{n} = \lim_{n\to\infty} \sum_{i=1}^n \left[-2 - \frac{6i}{n}\right]\frac{2}{n} = \lim_{n\to\infty} -\frac{2}{n}(2n+3(n+1)) = -10$$

(c) $\int_0^5 1 + 2x^3\, dx$

If we simplify the function first,

$$f(5i/n) = 1 + 2(5i/n)^3 = 1 + \frac{250}{n^3} i^3$$

Using the Riemann sum, if we sum this expression for $i = 1..n$, we get

$$\sum_{i=1}^{n} f(5i/n) = n + \frac{250}{n^3} \cdot \frac{n^2(n+1)^2}{4}$$

Multiply by $5/n$ and take the limit, and we get:

$$5 + \frac{625}{2} = \frac{635}{2}$$

5. Evaluate the integral and interpret it as the area of a region (sketch it).

$$\int_0^4 |\sqrt{x+2} - x|\, dx$$

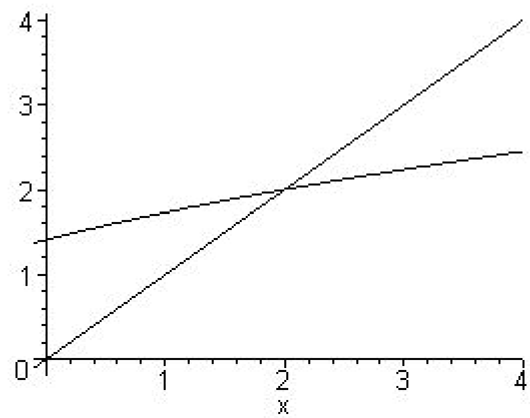
We see that the curves intersect where

$\sqrt{x+2} = x, \quad x^2 - x - 2 = 0, \quad x = -1, 2$

Using the picture, we see that the area is represented by the integral sum:

$$\int_0^2 \sqrt{x+2} - x\, dx + \int_2^4 x - \sqrt{x+2}\, dx =$$

$$\frac{10}{3} - \frac{4\sqrt{2}}{3} - 4\sqrt{6} + \frac{34}{3} \approx 2.983$$



6. True or False (and give a short reason):

(a) $\int_0^2 (x - x^3)\, dx$ represents the area under the curve $y = x - x^3$ from 0 to 2.
FALSE. The function is negative for $1 \le x \le 2$, so the integral represents the *net area* between the curve and the x axis. If you wanted the actual area, you would need to integrate $|x - x^3|$.

(b) If $3 \le f(x) \le 5$ for all x, then $6 \le \int_1^3 f(x)\, dx \le 10$
TRUE. We're using the property that, if $m \le f(x) \le M$, then $m(b-a) \le \int_a^b f(x)\, dx \le M(b-a)$.

(c) If f, g are continuous on $[a, b]$, then

$$\int_a^b f(x) - g(x)\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

TRUE. This is one of the properties of the definite integral.

(d) The fact that f, g were each individually continuous on $[a, b]$ was an important thing to state in the last problem.

TRUE. Otherwise, you could do something silly like:

$$\int_{-1}^{1} 1\, dx = \int_{-1}^{1} (1 + \frac{1}{x}) - \frac{1}{x}\, dx$$

and the integral of $1/x$ would not exist on this interval.

(e) If f, g are continuous on $[a, b]$, then

$$\int_a^b f(x)g(x)\, dx = \left(\int_a^b f(x)\, dx\right)\left(\int_a^b g(x)\, dx\right)$$

FALSE. For example, if $f(x) = 3$ and $g(x) = 1$, then the antiderivative of $f(x)g(x)$ is $3x$, but the product of antiderivatives would be $(3x)(x) = 3x^2$.

Hint: If you want to say that something is false, provide a quick counterexample.

(f) All continuous functions have derivatives.

FALSE. The famous example from Calc I is $y = |x|$ at $x = 1$. Continuity does not imply differentiability.

(g) All continuous functions have antiderivatives.

TRUE. This is what the Fundamental Theorem of Calculus says- $g(x) = \int_a^x f(t)\, dt$ is an antiderivative if f is continuous.

7. For each of the following Riemann sums, evaluate the limit by first recognizing it as an appropriate integral:

(a) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3}{n}\right) \sqrt{1 + \frac{3i}{n}}$ (Find four different integrals for this one!)

From what is given, we know that $b - a = 3$, so if a or b is given, the other can be computed. Since we see the expression

$$1 + \frac{3i}{n}$$

then we might go ahead and take $a = 1$ (so that b must be 4). In this case, the integral will be

$$\int_1^4 \sqrt{x}\, dx$$

However, a second choice would be to take $a = 0$, so that

$$f(3i/n) = \sqrt{1 + 3i/n} \quad \Rightarrow \quad f(x) = \sqrt{1 + x}$$

and the integral would be

$$\int_0^3 \sqrt{1 + x}\, dx$$

A third alternative: Let's take $a = 3$ just for fun. Then,

$$f(3 + 3i/n) = \sqrt{1 + 3i/n} = \sqrt{-2 + (3 + 3i/n)} \quad \Rightarrow \quad f(x) = \sqrt{x-2}$$

and the integral would be $\displaystyle\int_3^6 \sqrt{x-2}\,dx$.

As a last option, suppose $a = -1$. Then we would have

$$f(-1 + 3i/n) = \sqrt{1 + 3i/n} = \sqrt{2 + (-1 + 3i/n)} \quad \Rightarrow \quad f(x) = \sqrt{x+2}$$

and the integral would be $\displaystyle\int_{-1}^2 \sqrt{x+2}\,dx$

(b) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^n \left(2 + 3\cdot\frac{25i^2}{n^2}\right)\left(\frac{5}{n}\right)$

Some options:

$$\int_0^5 2 + 3x^2\,dx \qquad \int_1^6 2 + 3(x-1)^2\,dx \qquad \int_2^7 2 + 3(x-2)^2\,dx$$

8. Evaluate the integral, if it exists

(a) $\displaystyle\int_1^9 \frac{\sqrt{u} - 2u^2}{u}\,du$

SOLUTION: Use algebra first to simplify.

$$\int_1^9 u^{-1/2} - 2u\,du = 2u^{1/2} - u^2\Big|_1^9 = (2(3) - 81) - (2 - 1) = -76$$

(b) $\displaystyle\int 3^x + \frac{1}{x} + \sec^2(x)\,dx$

SOLUTION: These are an assortment of functions from the table:

$$\frac{1}{\ln(3)}3^x + \ln|x| + \tan(x) + C$$

(c) $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{t^4\tan(t)}{2+\cos(t)}\,dt$

SOLUTION: By symmetry (the function is odd, since $\tan(t)$ is odd, and is multiplied by an even function), the integral is zero.

(d) $\displaystyle\int_0^3 |x^2 - 4|\,dx$

SOLUTION: Break up the interval to get rid of the absolute value:

$$\int_0^2 -(x^2-4)\,dx + \int_2^3 x^2 - 4\,dx = \cdots = \frac{16}{3} + \frac{7}{3} = \frac{23}{3}$$

(e) $\displaystyle\int \frac{\cos(\ln(x))}{x}\,dx$

SOLUTION: Use $u = \ln(x)$, $du = 1/x\,dx$ so:

$$\int \frac{\cos(\ln(x))}{x}\,dx = \int \cos(u)\,du = \sin(u) + C = \sin(\ln(x)) + C$$

(f) $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$

SOLUTION: Use geometry to see this is the area of a quarter circle-

$$y = \sqrt{4-x^2} \quad \Rightarrow \quad x^2+y^2=4$$

so the area is π.

(g) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$

SOLUTION: The integrand is the derivative of $\sin^{-1}(x)$, so the answer is $\sin^{-1}(x)+C$.

(h) $\displaystyle\int_{-1}^2 \frac{1}{x}\,dx$

SOLUTION: The function is not continuous on the interval $[-1,2]$, so we would say that the FTC does not apply.

(i) $\displaystyle\int_0^1 (\sqrt[4]{w}+1)^2\,dw$

SOLUTION: Multiply it out first: $(w^{1/4}+1)^2 = w^{1/2}+2w^{1/4}+1$, so

$$\int w^{1/2}+2w^{1/4}+1\,dw = \frac{2}{3}w^{3/2}+\frac{8}{5}x^{5/4}+w+C$$

(j) $\displaystyle\int_{-2}^{-1} \frac{1}{x}\,dx = 0-\ln(2) = \ln(1/2)$

(Did you remember to use $\ln|x|$?)

(k) $\displaystyle\int_0^{1/2} \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dt$

SOLUTION: Let $u=\sin^{-1}(x)$, so $du = dx/\sqrt{1-x^2}$. Substitute, with $\sin^{-1}(0) = 0$ and $\sin^{-1}(1/2) = \pi/6$, since $\sin(\pi/6) = 1/2$:

$$\int_0^{\pi/6} u\,du = \left.\frac{1}{2}u^2\right|_0^{\pi/6} = \frac{\pi^2}{72}$$

(l) $\displaystyle\int (1+\tan(t))\sec^2(t)\,dt$

SOLUTION: Let $u=\tan(t)$, so $du = \sec^2(t)\,dt$, and the integral becomes

$$\int 1+u\,du = u+\frac{1}{2}u^2 = \tan(t)+\frac{1}{2}\tan^2(t)+C$$

(m) $\displaystyle\int \tan(x)\,dx$

SOLUTION: Re-write the integrand as $\sin(x)/\cos(x)$, then let $u=\cos(x)$. Therefore,

$$\int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{1}{u}\,du = -\ln|u|+C = -\ln|\cos(x)|+C = \ln|\sec(x)|+C$$

(Its OK if you don't do the last step).

(n) $\int x\sqrt{1+x}\,dx$

SOLUTION: Let $u = 1+x$. Then $du = dx$ and if we substitute now, we see there is an extra x. Go to the first equation, $u = 1+x$ and solve for x in terms of u: $x = 1-u$, and now we have:

$$\int (1-u)u^{1/2}\,du = \int u^{1/2} - u^{3/2}\,du = \frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2} + C = \frac{2}{3}(1+x)^{3/2} - \frac{2}{5}(1+x)^{5/2} + C$$

(o) $\int \frac{y-1}{\sqrt{3y^2-6y+4}}\,dy$

SOLUTION: Let $u = 3y^2 - 6y + 4$, so $du = 6y - 6\,dy = 6(y-1)\,dy$. Substitute into the equation and:

$$\frac{1}{6}\int u^{-1/2}\,du = \frac{1}{3}u^{1/2} + C = \frac{1}{3}(3y^2 - 6y + 4)^{1/2} + C$$

(p) $\int_{-1}^{4} |t-3|\,dt$

SOLUTION: Break up the integral

$$\int_{-1}^{3} 3 - t\,dt + \int_{3}^{4} t - 3\,dt = \frac{17}{2}$$

You can also do it using geometry and add the areas of the two triangles together.

9. Find the derivative of the function:

Note: For each of these, we're using the formula from the FTC part I

$$y = \int_{g(x)}^{h(x)} f(t)\,dt \quad \Rightarrow \quad y' = f(h(x))h'(x) - f(g(x))g'(x)$$

(a) $F(x) = \int_0^{x^2} \frac{\sqrt{t}}{1+t^2}\,dt$ so $F'(x) = \frac{2x^2}{1+x^4}$, assuming $x > 0$.

(b) $y = \int_{\sqrt{x}}^{3x} \frac{e^t}{t}\,dt$

$$y' = \frac{3e^{3x}}{3x} - \frac{1}{2\sqrt{x}}\frac{e^{\sqrt{x}}}{\sqrt{x}} = \frac{e^{3x}}{x} - \frac{e^{\sqrt{x}}}{2x}$$

10. The idea here was to recognize this as a Riemann Sum, then evaluate by evaluating the definite integral. For example,

$$\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \cdots \left(\frac{n}{n}\right)^9\right] = \int_0^1 x^9\,dx = \frac{1}{10}x^{10}\Big|_0^1 = \frac{1}{10}$$

11. Evaluate:

(a) $\int_0^1 \frac{d}{dx}\left(e^{\tan^{-1}(x)}\right)\,dx = e^{\tan^{-1}(1)} - e^{\tan^{-1}(0)} = e^{\pi/4} - 1$

Generally speaking, this is $\int_a^b f'(x)\,dx = f(b) - f(a)$.

(b) $\dfrac{d}{dx}\displaystyle\int_0^1 \mathrm{e}^{\tan^{-1}(x)}\,dx = 0$ (This is the derivative of a constant).

(c) $\dfrac{d}{dx}\displaystyle\int_0^x \mathrm{e}^{\tan^{-1}(t)}\,dt = \mathrm{e}^{\tan^{-1}(x)}$

12. (a) Sketch the graph of $f(x) = |x| - 1$.

 (b) Suppose this function is the derivative of some other function, $F(x)$. Sketch one possibility using your previous graph as a guide.

 (c) Sketch the function $G(x) = \int_{-2}^{x} f(t)\,dt$ for the same values of $-2 \le x \le 4$, again using your previous answers as a guide.

 (d) What is the relationship (if any) between F and G?

 SOLUTION: Please review how to draw the sketch of an antiderivative from the graph of the function (or equivalently, given the graph of f', sketch f). The sketches from (b) and (c) are in Figure 1. Are you curious about how these were plotted? The antiderivative of $|x|$ is actually $\frac{1}{2}x|x|$. For fun, see if you can show it.

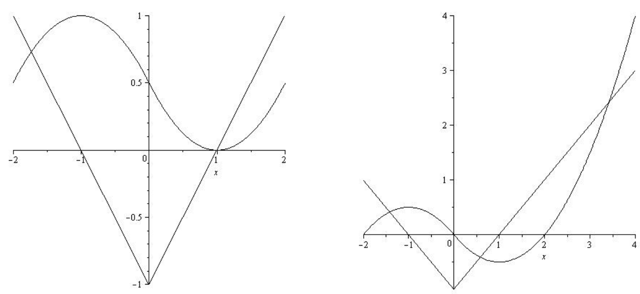


Figure 1: Solutions to Exercise 12. The figure to the left is a generic antiderivative. The figure to the right is the antiderivative that is zero at $x = -2$. One is just a constant shift of the other.

13. A particle moves along a line with velocity $v(t) = t^2 - t$, where v is measured in meters per second. Find (a) the displacement and (b) the distance traveled by the particle during the time interval $[0, 5]$.

 SOLUTION: The displacement will simply be the integral:

$$\int_0^5 t^2 - t\,dt = \left(\frac{1}{3}t^3 - \frac{1}{2}t^2\right|_0^5 = \frac{175}{6} \approx 29.17$$

 Sorry about the fractions- I'll try to keep the numbers somewhat nice for the exam.

 The distance traveled is the absolute value of the velocity. Notice that the velocity function is an upward opening parabola with zeros at 0 and 1. Therefore,

$$\int_0^5 |t^2 - t|\,dt = \int_0^1 -t^2 + t\,dt + \int_1^5 t^2 - t\,dt = \frac{1}{6} + \frac{88}{3} = \frac{59}{2} = 29.5$$

14. If f is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 x f(x^2)\,dx$

SOLUTION: Use u, du substitution with $u = x^2$, so $du = 2x\,dx$, and

$$\int_0^3 x f(x^2)\,dx = \frac{1}{2}\int_0^9 f(u)\,du = \frac{1}{2}\cdot 4 = 2$$

15. If $f''(x) = 2 - 12x$, $f(0) = 0$ and $f(2) = 15$, find $f(x)$.

SOLUTION: $f'(x) = 2x - 6x^2 + C_1$ so $f(x) = x^2 - 2x^3 + C_1 x + C_2$. We use the information provided to solve for the constants:

$$f(0) = 0 \quad \Rightarrow \quad 0 + 0 + 0C_1 + C_2 = 0$$

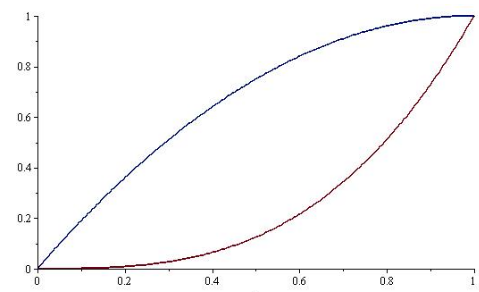
so

$$f(2) = 15 \quad \Rightarrow 2^2 - 22^3 + 2C_1 = 15 \quad C_1 = \frac{27}{2}$$

Therefore

$$f(x) = x^2 - 2x^3 + \frac{27}{2}x$$

16. Let R be the region in the first quadrant bounded by $y = x^3$ and $y = 2x - x^2$. Calculate the following quantities: (Exam note: Region R would typically be plotted for you).



(a) The area of R.

$$A = \int_0^1 (2x - x^2) - x^3\,dx = \frac{5}{12}$$

(b) Volume obtained by rotating R about the $x-$axis.

SOLUTION: Using washers, the inner radius is $r = x^3$ and the outer radius is $R = 2x - x^2$, to the volume is:

$$V = \int_0^1 \pi[(2x - x^2)^2 - (x^3)^2]\,dx = \pi\int_0^1 [4x^2 - 4x^3 + x^4 - x^6]\,dx =$$

$$\pi\left[\frac{4}{3} - 1 + \frac{1}{5} - \frac{1}{7}\right] = \frac{41}{105}\pi$$

(c) Volume obtained by rotating R about the $y-$axis.

SOLUTION: Use shells so we can stay in terms of x: The radius is x and the height is $(2x - x^2) - x^3$, so

$$V = \int_0^1 2\pi x(2x - x^2 - x^3)\,dx = \frac{13}{30}\pi$$

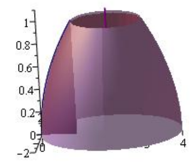
17. Use any method to find an integral representing the volume generated by rotating the given region about the specified axis. You do NOT need to evaluate the integral:

(a) $y = \sqrt{x}, y = 0, x = 1$; about $x = 2$.
In terms of shells, the radius is $2 - x$ and the height is $\sqrt{x}$:

$$V = \int_0^1 2\pi(2 - x)\sqrt{x}\, dx$$

In terms of washers, the outer radius is $2 - y^2$ and the inner radius is 1:

$$V = \int_0^1 \pi((2 - y^2)^2 - 1^2)\, dy$$

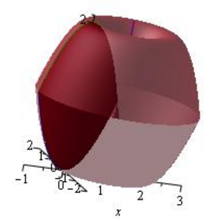


The solid of revolution created on $0 \le x \le 1$ by rotation of $f(x) = \sqrt{x}$ about the axis $x = 2$. The slice that is rotated is shaded in burgundy.

(b) $y = x^2$, $y = 2 - x^2$; about $x = 1$.

In terms of shells, the radius is $1 - x$ and the height is $(2 - x^2) - x^2$:

$$V = \int_{-1}^1 2\pi(1 - x)(2 - 2x^2)\, dx$$

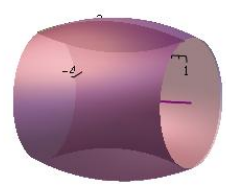


The solid of revolution created on $-1 \le x \le 1$ by rotation of $f(x) = x^2$ and $g(x) = 2 - x^2$ about the axis $x = 1$. The slice that is rotated is shaded in burgundy.

(c) $y = x^2$, $y = 2 - x^2$; about $y = -3$.

In terms of washers (preferred method), the inner radius is $x^2 - (-3) = x^2 + 3$ and the outer radius is $(2 - x^2) - (-3) = 5 - x^2$:

$$V = \int_{-1}^1 \pi\left[(5 - x^2)^2 - (x^2 + 3)^2\right]\, dx$$

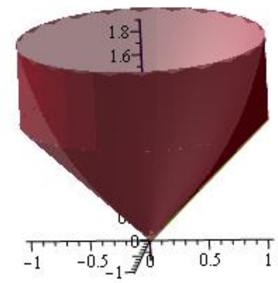


The solid of revolution created on $-1 \le x \le 1$ by rotation of $f(x) = x^2$ and $g(x) = 2 - x^2$ about the axis $y = -3$. The slice that is rotated is shaded in burgundy.

(d) $y = \tan(x)$, $y = x$, $x = \pi/3$; about the $y-$axis.

In terms of shells, the radius is x and the height is $\tan(x) - x$:

$$V = \int_0^{\pi/3} 2\pi x(\tan(x) - x)\, dx$$



The solid of revolution created on $0 \le x \le \frac{\pi}{3}$ by rotation of $f(x) = \tan(x)$ and $g(x) = x$ about the axis $x = 0$. The slice that is rotated is shaded in burgundy.

18. Prove the following using induction:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

SOLUTION: Using induction, we:

- Prove true for $n = 1$ (or some starting case):
$$1^2 = \frac{1(2)(3)}{6} \text{ Yes.}$$
- Assume true for $n = k$:
$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$
Now use the assumption to prove that the statement must be true for $n = k+1$, or, that
$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$
We start with the left side of the equation, and stop when we show the right side of the equation:
$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 =$$
$$(k+1)\frac{k(2k+1) + 6k + 6}{6} = (k+1)\frac{2k^2 + 7k + 6}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

19. Prove the following using induction:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

SOLUTION: Proof:

- We first prove that the statement is true if $n = 1$. In this case, statement becomes: $1/2 = 1/2$, which is true.
- We assume that the statement is true if $n = k$. That is,

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}.$$

- We show, using our assumption, that the statement must be true when $n = k+1$. That is,

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}.$$

  We do that by starting with the LHS of the equation, then showing that we can get the RHS:

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} && \text{Break apart the sum} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} && \text{By assumption} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{k+1}{k+2} && \text{QED}
\end{aligned}$$

20. Use a series to evaluate the following limit: $\displaystyle\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$

SOLUTION: We want to use the Maclaurin series for $\sin(x)$:

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots$$

Substitute this into the expression:

$$\frac{\sin(x) - x}{x^3} = \frac{-\frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots}{x^3} = -\frac{1}{3!} + \frac{1}{5!}x^2 + \cdots$$

The limit is $-1/6$.

21. Use a known template series to find a series for the following:

(a) $\frac{x^2}{1+x}$

SOLUTION: You should be thinking about the sum of a geometric series, $1/(1-r)$. We can put it into that form:

$$\frac{x^2}{1+x} = x^2 \cdot \frac{1}{1-(-x)} = x^2 \cdot \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+2} = \sum_{m=2}^{\infty} x^m$$

You don't need to do the last step, but it may be helpful.

(b) 10^x

SOLUTION: Using the hint and the Maclaurin series for the exponential function:

$$10^x = \mathrm{e}^{\ln(10^x)} = \mathrm{e}^{x\ln(10)}$$

Now, the Maclaurin series for the exponential:

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

so that

$$10^x = \sum_{n=0}^{\infty} \frac{(x\ln(10))^n}{n!}$$

(c) $x\mathrm{e}^{2x}$

SOLUTION: Use the Maclaurin series for e^x:

$$x\mathrm{e}^{2x} = x\sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n!}$$

22. Find the Taylor series for $f(x)$ centered at the given base point:

(a) $x^4 - 3x^2 + 1$, at $x = 1$

SOLUTION: We don't really need to build a table for this one, but we can:

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^4 - 3x^2 + 1$	-1
1	$4x^3 - 6x$	-2
2	$12x^2 - 6$	6
3	$24x$	24
4	24	24
5	0	0
$\vdots$		$\vdots$

We see that we get:

$$f(x) = -1 - 2(x-1) + \frac{6}{2!}(x-1)^2 + \frac{24}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4$$

We should pause and note that if we were to expand this expression and simplify, it would get us back to $x^4 - 3x^2 + 1$. We would also note that this is a finite sum and NOT an infinite series, so that a "radius of convergence" is not really needed- the sum exists for all x.

(b) $1/\sqrt{x}$ at $x = 9$ (just get the first four non-zero terms of the power series).

n	$f^{(n)}(x)$	$f^{(n)}(9)$
0	$x^{-1/2}$	$\frac{1}{2^0 3}$
1	$(-1/2)x^{-3/2}$	$\frac{-1}{2^1 3^3}$
2	$(-1/2)(-3/2)x^{-5/2}$	$\frac{1\cdot 3}{2^2 3^5}$
3	$(-1/2)(-3/2)(-5/2)x^{-7/2}$	$\frac{-1\cdot 3\cdot 5}{2^3 3^7}$

If we look that over carefully, we see the general term:

$$f^{(n)}(9) = \frac{(-1)^n (1\cdot 3\cdot 5 \cdots (2n-1))}{2^n 3^{2n+1}}$$

If you didn't get the general term, that's fine- We were asked to write the first four terms:

$$\frac{1}{3} - \frac{1}{2\cdot 3^3}(x-9) + \frac{1\cdot 3}{2!2^2 3^5}(x-9)^2 - \frac{1\cdot 3\cdot 5}{3!2^3 3^7}(x-9)^3 + \cdots$$

(c) x^{-2} at $x = 1$. In this case, find a pattern for the n^{th} coefficient so that you can write the general series. Using this answer, find the radius of convergence.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-2}	1
1	$-2x^{-3}$	-2
2	$(-2)(-3)x^{-4}$	$3!$
3	$(-2)(-3)(-4)x^{-5}$	$-4!$

$$\Rightarrow \quad f^{(n)}(1) = (-1)^n (n+1)!$$

Putting this into the Maclaurin series, we get the series that we'll test:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{n!}(x-1)^n = \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$$

Now perform the Ratio Test to see that the radius of convergence is 1.

23. Find the Maclaurin series and radius of convergence for $\ln(x+1)$.

SOLUTION: This is much like the last one. First come up with a formula for $f^{(n)}(0)$:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(x+1)$	0
1	$(x+1)^{-1}$	1
2	$(-1)(x+1)^{-2}$	-1
3	$(-1)(-2)(x+1)^{-3}$	2
4	$(-1)(-2)(-3)(x+1)^{-4}$	$-3!$

$$\Rightarrow \quad f^{(n)}() = (-1)^{n+1}(n-1)!$$

Notice that n starts with 1 instead of 0. Substitution gives us:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

and the Ratio Test should give you a radius of 1.