Solutions to the Review Questions, Exam 3

1. Write the partial fraction decomposition for each of the following (do not actually solve for the coefficients):

   (a) $\dfrac{3-4x^2}{(2x+1)^3} = \dfrac{A}{2x+1} + \dfrac{B}{(2x+1)^2} + \dfrac{C}{(2x+1)^3}$

   (b) $\dfrac{7x-41}{(x-1)^2(2-x)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{2-x}$

   (c) $\dfrac{x+1}{x^3(x^2-x+10)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{Dx+E}{x^2-x+10} + \dfrac{Fx+G}{(x^2-x+10)^2}$

   We note that $x^2 - x + 10$ is irreducible, since $b^2 - 4ac = 1 - 4 \cdot 10 < 0$.

2. Suppose I made the substitution $x = \tan(\theta)$, and after integration I got the expression $\theta + \sin(2\theta)$. Convert it back to x.

   SOLUTION: From the substitution, we note two things-

   $$\theta = \tan^{-1}(x)$$

   and the substitution gives the relationship on a triangle, where if θ is an angle, then x is the length of the side opposite and 1 is the length of the side adjacent, so the hypotenuse has length $\sqrt{1+x^2}$.

   We can't get $\sin(2\theta)$ directly, but we can use the identity:

   $$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2 \cdot \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}}$$

   Overall, the solution is:

   $$\tan^{-1}(x) + \frac{2x}{1+x^2}$$

   NOTE: The expression $\sin(2\tan^{-1}(x))$ *is not a simplified expression in* x*, so is not a complete answer to the problem.*

3. Same kind of problem as the previous one, but $t + 2 = \sqrt{3}\sec(\theta)$, and the result was $\cos(2\theta)$ (TYPO: The question should read "convert this expression back to t").

   SOLUTION: We get the triangle from the substitution, and we need to recall the identity for $\cos(2\theta)$, of which there are several possibilities. If you've forgotten them, hopefully you recall the half angle formula:

   $$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2} \qquad \Rightarrow \qquad \cos(2\theta) = 2\cos^2(\theta) - 1$$

   Now subsititute the appropriate expression in for $\cos(\theta)$:

   $$2\left(\frac{\sqrt{3}}{t+2}\right)^2 - 1 = \frac{6}{(t+2)^2} - 1$$

4. (a) The substitution would be $x = 2\sin(\theta)$, so that the integrand becomes:

$$\frac{4\sin^2(\theta)\,2\cos(\theta)}{(4-x^2)^{3/2}} = \frac{8\sin^2(\theta)\cos(\theta)}{(4(1-\sin^2(\theta)))^{3/2}} = \frac{8\sin^2(\theta)\cos(\theta)}{8\cos^3(\theta)} = \tan^2(\theta)$$

Therefore, we get

$$\int \frac{x^2}{(4-x^2)^{3/2}}\,dx = \int \tan^2(\theta)\,d\theta$$

(b) For the second integral, normally we would first check to see if the quadratic is reducible (it is not; $b^2 - 4ac < 0$). However, in this case, we're told to make the trig substitution- "Complete the square" in the denominator:

$$\frac{x}{x^2+2x+5} = \frac{x}{(x^2+2x+1)+4} = \frac{x}{(x+1)^2+4}$$

We take $x + 1 = 2\tan(\theta)$ so that the integrand becomes:

$$\int \frac{(2\tan(\theta)-1)2\sec^2(\theta)\,d\theta}{4\tan^2(\theta)+4} = \int \frac{(2\tan(\theta)-1)\,d\theta}{2}$$

5. Show that $\int x f''(x)\,dx = x f'(x) - f(x)$

This is integration by parts:

$$\begin{array}{ccc} + & x & f''(x) \\ - & 1 & f'(x) \\ + & 0 & f(x) \end{array} \quad \Rightarrow \quad x f'(x) - f(x)$$

6. For any spring obeying Hooke's law, show that the work done in stretching a spring a distance of d units (past natural length) is given by $W = \frac{1}{2}kd^2$

SOLUTION: From what is given:

$$W = \int_0^d kx\,dx = \frac{1}{2}kx^2\Big|_0^d = \frac{1}{2}kd^2$$

7. Set up (do not compute) the integral representing the work done pumping the water over the rim of a tank that is 6 meters long and has a semicircular end of radius 4 meters (the bottom half of a circle). The tank is filled to a depth of 3 meters, and assume the density of water is given by σ kg/m^3 (where σ is approximately constant), and $g = 9.8$m/s^2.

SOLUTION: We'll set up a coordinate system so that the tank is the bottom half of a circle of radius 4, and the top of the tank is the $x-$axis. Using this, then y ranges from -4 to -1, and if we take a slab or slice of water, the width of the slab is $2x$, where $x = \sqrt{16 - y^2}$. With gravity g and density σ, the integrand is given by the following (checking units to be units of force, or kg·m/s^2.

$$\text{width} \times \text{length} \times \text{distance} \times \text{density} \times \text{accel}$$

$$2\sqrt{16-y^2}\ \text{m} \times 6\ \text{m} \times (-y)\ \text{m} \times \sigma\frac{\text{kg}}{\text{m}^3} \times g\frac{\text{m}}{\text{s}^2}$$

$$12\sigma g \int_{-4}^{-1} \sqrt{16-y^2}(-y)\,dy = 12\sigma g \int_{-1}^{-4} y\sqrt{16-y^2}\,dy$$

If we take $u = 16 - y^2$, then the integral becomes the following (change the bounds, too):

$$-6\sigma g \int_{15}^{0} \sqrt{u}\,du = 6\sigma g 15^{3/2}$$

The $15^{3/2}$ term doesn't simplify any further. We can leave our answer in this form.

8. A 10 lb monkey hangs at the end of a 20 ft chain weighing 1/2 lb per ft. How much work is done by the monkey in climbing up the chain, if the chain is attached to the monkey?

   Hint: You might consider the work using y as the position of the bottom of the chain, so that $0 \le y \le 10$.

   SOLUTION: We'll break this into two parts- One for the monkey, the other for the chain.

   For the monkey, the force is constant at 10 lbs, so the work is $10 \times 20 = 200$.

   For the chain, first set y as the distance from the bottom of the chain to the floor (so that y ranges from 0 to 10). The monkey will have $20 - 2y$ feet to climb (draw a sketch to see this), carrying a slice of chain that weighs $\frac{1}{2}\Delta y$ pounds. Therefore, the work for just the chain will be given by

$$\int_0^{10} \frac{1}{2}(20-2y)\,dy = 10y - \frac{1}{2}y^2\Big|_0^{10} = 50 \text{ ft-lbs}$$

   Overall, the work involved will be 250 ft-lbs.

9. True or False? (And give a short reason)

   (a) If f is continuous on $[0,\infty)$ and $\int_1^\infty f(x)\,dx$ converges, then $\int_0^\infty f(x)\,dx$ converges.
   SOLUTION: This is true, since we can write:

$$\int_0^\infty f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx$$

   The first part of the sum (on the interval $[0,1]$) exists because we're told that f is continuous on $[0,\infty)$, and therefore f is continuous on $[0,1]$ so that the Fundamental Theorem of Calculus (part 1) is satisfied there.

   (b) To find $\int \sin^2(x)\cos^5(x)\,dx$, rewrite the integrand as $\sin^2(x)(1-\sin^2(x))^2\cos(x)$
   SOLUTION: That is true; then let $u = \sin(x)$ and $du = \cos(x)\,dx$.

   (c) Integration by parts is the integral version of the Product Rule for derivatives.

   SOLUTION: That is true. We showed it in class, but you could also start with the product rule, then integrate both sides:

$$(fg)' = f'g + fg' \quad \to \quad f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

(d) To find $\int \frac{2x-3}{x^2-3x+5}\,dx$, start by completing the square in the denominator.
SOLUTION: False- Start by looking for an obvious u, du substitution- In this case, $u = x^2 - 3x + 5$ and $du = 2x - 3\,dx$

Side Note: Would it be wrong to complete the square? You would get to the same answer, but it would take you significantly more time...

(e) To find $\int \frac{3}{x^2-3x+5}\,dx$, start by completing the square in the denominator.
SOLUTION: False. Start by checking that you cannot factor the denominator- In this case, we cannot, so then continue by completing the square.

(f) To find $\int \frac{3}{x^2-4x+3}\,dx$, start by completing the square in the denominator.
SOLUTION: False. Start by checking the denominator- In this case, we can factor it, so we should do that and use partial fractions.

(g) u, du substitution is the integral version of the Chain Rule.
SOLUTION: True. We showed it in class, and gives you some good insight into when to use it.

10. Does the following integral converge or diverge? $\displaystyle\int_1^\infty \frac{2+\sin(x)}{\sqrt{x}}\,dx$

SOLUTION: Since $-1 \le \sin(x) \le 1$, then $1 \le 2+\sin(x) \le 3$, so that

$$0 \le \frac{1}{\sqrt{x}} \le \frac{2+\sin(x)}{\sqrt{x}}$$

Further,

$$\int_1^\infty x^{-1/2}\,dx = \lim_{t\to\infty} (2\sqrt{x}|_1^t = \lim_{t\to\infty} 2\sqrt{t} - 2 \to \infty$$

Therefore, by the comparison theorem, the original integral also diverges.

11. Does the integral converge or diverge? If it converges, evaluate it.

(a) $\displaystyle\int_0^\infty t\mathrm{e}^{-st}\,dt$
(s is a constant- state any conditions on s for the integral to converge.)
SOLUTION: First we'll take care of the integration. Use integration by parts, we get the following (I've put it into a single fraction, but that is not necessary):

$$\begin{array}{c|c|c} + & t & \mathrm{e}^{-st} \\ - & 1 & (-1/s)\mathrm{e}^{-st} \\ + & 0 & (1/s^2)\mathrm{e}^{-st} \end{array} \quad \Rightarrow \quad -\left.\frac{(st+1)\mathrm{e}^{-st}}{s^2}\right|_0^T = \lim_{T\to\infty} -\frac{(sT+1)\mathrm{e}^{-sT}}{s^2} + \frac{1}{s^2}$$

For the limit, we can factor out the s^2 (it's constant with respect to T), and we get a fraction on which we can use l'Hospital's rule:

$$\frac{-1}{s^2}\lim_{T\to\infty}\frac{sT+1}{\mathrm{e}^{sT}} = \frac{-1}{s^2}\lim_{T\to\infty}\frac{s}{s\mathrm{e}^{sT}} = 0$$

The previous steps were valid as long as $s > 0$ (otherwise, e^{-sT} would diverge to $-\infty$). Overall then, the integral converges to $1/s^2$.

(b) $\displaystyle\int_1^4 \frac{dx}{\sqrt{x-1}}$

SOLUTION: Rewriting the integrand as $(x-1)^{-1/2}$, we see that the antiderivative is $2(x-1)^{1/2} = 2\sqrt{x-1}$. Therefore,

$$\int_1^4 \frac{dx}{\sqrt{x-1}} = \lim_{t\to 1^+} 2\sqrt{x-1}\Big|_t^4 = \lim_{t\to 1^+}\left(2\sqrt{3} - 2\sqrt{t-1}\right) = 2\sqrt{3}$$

(c) $\displaystyle\int_3^\infty \frac{\ln(x)}{x}\,dx$

SOLUTION: The integral diverges. We can use the comparison theorem with $1/x$. That is, since $1 < \ln(x)$ for $x > e$, then for $x > 3$, we have:

$$\frac{1}{x} < \frac{\ln x}{x}$$

and $\int_3^\infty 1/x\,dx$ diverges.

ALTERNATIVE SOLUTION: You could perform the integration, and show that the limit diverges. In this case, if we take care of the integrand first:

$$\int \frac{\ln(x)}{x}\,dx \qquad \begin{array}{c} u = \ln(x) \\ du = (1/x)\,dx \end{array} \qquad \int u\,du = \frac{1}{2}u^2 = \frac{1}{2}(\ln(x))^2$$

Now we can take the limit:

$$\int_3^\infty \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty} \frac{1}{2}(\ln(x))^2\Big|_3^t$$

This limit diverges to infinity.

(d) $\displaystyle\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$

SOLUTION: Rewrite the integral using a convenient number:

$$\int_{-\infty}^0 \frac{x}{x^2+1}\,dx + \int_0^\infty \frac{x}{x^2+1}\,dx$$

For each, we can use $u = x^2+1$ and $\frac{1}{2}du = x\,dx$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln(x^2+1)$$

This expression diverges if $x \to \infty$ and when $x \to -\infty$, so the integral diverges.

12. Evaluate using any method, unless specified below:

(a)

$$\int \frac{4\,dx}{(4+x^2)^{3/2}}$$

SOLUTION: Trig substitution is the most direct choice.

Let $x = 2\tan(\theta)$. Then

$$4 + x^2 = 4 + 4\tan^2(\theta) = 4\sec^2(\theta) \qquad \text{and} \qquad dx = 2\sec^2(\theta)\, d\theta$$

Substituting these in, we get:

$$\int \frac{8\sec^2(\theta)\, d\theta}{8\sec^3(\theta)} = \int \cos(\theta)\, d\theta = \sin(\theta) + C$$

Use the reference triangle to convert this answer back to x:

$$\frac{x}{\sqrt{4 + x^2}} + C$$

(b) $\displaystyle\int \tan^3(x)\sec^2(x)\, dx$

SOLUTION: This is a trig integral- Try to reserve something to pull off a u, du substitution. In this case, reserve $\sec^2(x)$ so that $u = \tan(x)$ and $du = \sec^2(x)\, dx$, and the integral becomes $\int u^3\, du$.

$$\frac{1}{4}\tan^4(x) + C$$

(c) $\displaystyle\int \frac{3x+2}{x^2+6x+8}\, dx = \int \frac{3x+2}{(x+2)(x+4)}\, dx$

SOLUTION: Since the denominator factors, use partial fractions. Here is the final answer:

$$= \int \frac{5}{x+4} - \frac{2}{x+2}\, dx = 5\ln|x+4| - 2\ln|x+2| + C$$

(d) $\displaystyle\int \frac{t^2\cos(t^3-2)}{\sin^2(t^3-2)}\, dt$

SOLUTION: Look for the u, du substitution first. In this case, we do have what we need, if we let $u = \sin(t^3 - 2)$. Then the integral becomes

$$\frac{1}{3}\int u^{-2}\, du = -\frac{1}{3}\csc(t^3-2) + C$$

(e) $\displaystyle\int \cos^5(x)\sqrt{\sin(x)}\, dx$

SOLUTION: Look for a substitution first. Looks like we can reserve one of the cosines for the du term, and make $u = \sin(x)$:

$$\int \cos^4(x)\sqrt{\sin(x)}\,[\cos(x)\, dx] = \int (1 - \sin^2(x))^2\sqrt{\sin(x)}\,[\cos(x)\, dx] =$$

$$\int (1-u^2)^2\sqrt{u}\, du = \int u^{1/2} - 2u^{5/2} + u^{9/2}\, du = \frac{2}{3}u^{3/2} - \frac{4}{7}u^{7/2} + \frac{2}{11}u^{11/2}$$

To finish up the problem, back substitute the x.

(f) $\displaystyle\int \frac{x}{x^2+4}\,dx$

SOLUTION: Straight u, du substitution: $\frac{1}{2}\ln|x^2+4| + C$.

(g) $\displaystyle\int \frac{dx}{\sqrt{1-6x-x^2}}$

SOLUTION: We'll need to complete the square in the denominator, then probably do a trig substitution. To complete the square, notice that

$$1-6x-x^2 = 1-(x^2+6x+\quad) = 10-(x+3)^2 = \sqrt{10}^2-(x+3)^2$$

I can make the substitution: $x+3=\sqrt{10}\sin(\theta)$ so that the denominator becomes:

$$\sqrt{10-10\sin^2(\theta)} = \sqrt{10}\cos(\theta)$$

and don't forget the dx term: $dx = \sqrt{10}\cos(\theta)\,d\theta$:

$$\int \frac{dx}{\sqrt{1-6x-x^2}} = \int \frac{\sqrt{10}\cos(\theta)\,d\theta}{\sqrt{10}\cos(\theta)} = \theta + C$$

Convert back to x to get

$$\sin^{-1}\left(\frac{x+3}{\sqrt{10}}\right) + C$$

(h) $\displaystyle\int \frac{x-1}{x^2+3}\,dx$

SOLUTION: It might be easiest to separate these into two integrals, or you could do a trig substitution. Separating we get:

$$\int \frac{x-1}{x^2+3}\,dx = \int \frac{x}{x^2+3}\,dx - \int \frac{1}{x^2+3}\,dx$$

The first integral is set up for u, du substitution. For the second integral, factor 3 from the denominator so that we can do a different u, du substitution:

$$\int \frac{1}{x^2+3}\,dx = \frac{1}{3}\int \frac{dx}{\left(\frac{x}{\sqrt{3}}\right)^2+1} = \frac{1}{\sqrt{3}}\int \frac{1}{u^2+1}\,du = \frac{1}{\sqrt{3}}\tan^{-1}(u)$$

Put the two together: $\frac{1}{2}\ln|x^2+3| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$

(i) $\displaystyle\int \sin^2(3t)\,dt$

SOLUTION: Use the half angle identity:

$$\int \sin^2(3t)\,dt = \frac{1}{2}\int 1-\cos(6t)\,dt = \frac{1}{2}t - \frac{1}{12}\sin(6t) + C$$

(j) $\displaystyle\int \frac{3x-2}{(x^2+2)^2}\,dx$

SOLUTION: We could break this into two, then use u, du substitution on one and trig substitution on the other, or we can just go for the trig substitution gusto from the start!

Let $x = \sqrt{2}\tan(\theta)$ and make the necessary substitutions to get:

$$\int \frac{3x-2}{(x^2+2)^2} = \int \frac{(3\sqrt{2}\tan(\theta)-2)(\sqrt{2}\sec^2(\theta))}{4\sec^4(\theta)}\,d\theta = \frac{\sqrt{2}}{4}\int \frac{(3\sqrt{2}\tan(\theta)-2)}{\sec^2(\theta)}d\theta$$

Continuing to simplify,

$$\frac{3}{2}\int \sin(\theta)\cos(\theta)\,d\theta - \frac{\sqrt{2}}{2}\int \cos^2(\theta)\,d\theta = \frac{3}{2}\int \sin(\theta)\cos(\theta)\,d\theta - \frac{\sqrt{2}}{4}\int (1+\cos(2\theta))\,d\theta$$

These can now each be evaluated to get:

$$\frac{3}{4}\sin^2(\theta) - \frac{\sqrt{2}}{4}\theta - \frac{\sqrt{2}}{8}\sin(2\theta) = \frac{3}{4}\sin^2(\theta) - \frac{\sqrt{2}}{4}\theta - \frac{\sqrt{2}}{4}\sin(\theta)\cos(\theta)$$

Finally, back substitute x using a triangle (which is why we converted $\sin(2\theta)$ in the previous answer). Unsimplified, the answer is:

$$\frac{3}{4}\left(\frac{x}{\sqrt{x^2+2}}\right)^2 - \frac{\sqrt{2}}{4}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{\sqrt{2}}{4}\frac{x}{\sqrt{x^2+2}}\cdot\frac{\sqrt{2}}{\sqrt{x^2+2}} + C$$

NOTE: If you evaluate $\int \sin(\theta)\cos(\theta)\,d\theta = -\frac{1}{2}\cos^2(\theta)$, you get a slightly different answer...

(k) $\int \sin^{-1}(x)\,dx$

Use integration by parts

$$\begin{array}{ccc} + & \sin^{-1}(x) & 1 \\ - & \frac{1}{\sqrt{1-x^2}} & x \end{array} \quad \Rightarrow \quad x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

Let $u = 1-x^2$, $du = -2x\,dx$ to finish: $x\sin^{-1}(x) + \sqrt{1-x^2} + C$

(l) $\int x^3\sqrt{x^2+4}\,dx$

Substitution: $u = x^2+4$, $du = 2x\,dx$, and $x^2 = u-4$. Then

$$\int x^3\sqrt{x^2+4}\,dx = \frac{1}{2}\int (u-4)u^{1/2}\,du = \frac{1}{2}\int u^{3/2} - 4u^{1/2}\,du$$

(and continue...)

$$\frac{1}{5}(x^2+4)^{5/2} - \frac{4}{3}(x^2+4)^{3/2} + C$$

(m) $\int \sqrt{2x-x^2}\,dx$

Complete the square first: $\int \sqrt{-(x^2-2x+1)+1}\,dx = \int \sqrt{1-(x-1)^2}\,dx$ Use a trig substitution: $\sin(\theta) = x-1$ and $\cos(\theta)\,d\theta = dx$. The integral becomes the following, which we can evaluate using either the half angle formulas or your table of formulas:

$$\int \cos^2(\theta)\,d\theta = \frac{1}{2}\cos(\theta)\sin(\theta) + \frac{1}{2}\theta$$

Use the reference triangle to convert back to x:

$$\frac{1}{2}(\sin^{-1}(x-1) + (x-1)\sqrt{2x-x^2}) + C$$

(n) $\int \sqrt{t}\ln(t)\,dt$

Integration by parts:

$$\begin{array}{ccc} + & \ln(t) & \sqrt{t} \\ - & 1/t & \frac{2}{3}t^{3/2} \end{array} \qquad \Rightarrow \qquad \frac{2}{3}t^{3/2}\ln(t) - \frac{2}{3}\int t^{1/2}\,dt = \frac{2}{3}t^{3/2}\ln(t) - \frac{4}{9}t^{3/2} + C$$

(o) $\int \frac{3x-1}{(x+2)(x-3)}\,dx$

By partial fractions,

$$\frac{3x-1}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3} \qquad \Rightarrow \qquad 3x-1 = A(x-3) + B(x+2)$$

Substitute $x=3$ to get $A = 7/5$ and substitute $x = -2$ to get $B = 8/5$. Then the integral becomes:

$$\int \frac{3x-1}{(x+2)(x-3)}\,dx = \frac{7}{5}\int \frac{1}{x+2}\,dx + \frac{8}{5}\int \frac{1}{x-3}\,dx = \frac{7}{5}\ln|x+2| + \frac{8}{5}\ln|x-3| + C$$

(p) $\int \ln(y^2+9)\,dy$

SOLUTION: Just like the regular log, we can integrate by parts

$$\begin{array}{c|c|c} + & \ln(y^2+9) & 1 \\ \hline - & \frac{2y}{y^2+9} & y \end{array} \qquad \Rightarrow \qquad y\ln(y^2+9) - 2\int \frac{y^2}{y^2+9}\,dy$$

For the integral in y, we can use trig substitution: $y = 3\tan(\theta)$ so that $y^2 + 9 = 9(\tan^2(\theta)+1) = 9\sec^2(\theta)$ and $dy = 3\sec^2(\theta)$:

$$\int \frac{y^2}{y^2+9}\,dy = \int \frac{9\tan^2(\theta)(3\sec^2(\theta))}{9\sec^2(\theta)}\,d\theta = 3\int \tan^2(\theta)\,d\theta$$

Now, use the formulas that will be handed out (about half way down the page) to get that

$$3\int \tan^2(\theta)d\theta = 3\left(\tan(\theta) - \theta\right)$$

Convert back to y so that:

$$-2\int \frac{y^2}{y^2+9}\,dy = -6\cdot\frac{y}{3} + 6\tan^{-1}\left(\frac{y}{3}\right)$$

Put it all together:

$$y\ln(y^2+9) - 2y + 6\tan^{-1}(y/3) + C$$

Alternative To compute $\int \frac{y^2}{y^2+9}\,dy$, we can also avoid the trig substitution by a couple of "tricks":

$$\int \frac{(y^2+9)-9}{y^2+9}\,dy = \int 1 - \frac{9}{y^2+9}\,dy = \int 1 - \frac{9}{9}\,\frac{1}{(y/3)^2+1}\,dy$$

$$\int 1\,dy - \int \frac{3\,du}{u^2+1} = y - 3\tan^{-1}(u) = y - 3\tan^{-1}(y/3)$$

(q) $\displaystyle\int \frac{\sin^3(x)}{\cos^4(x)}\,dx$

Retain one $\sin(x)$ to go with dx, and set up the substitution $u = \cos(x)$ $du = -\sin(x)\,dx$:

$$-\int (1-u^2)u^{-4}\,du = -\int u^{-4} - u^{-2}\,du = \frac{1}{3}\sec^3(x) - \sec(x) + C$$

(r) $\displaystyle\int \mathrm{e}^{-x}\sin(2x)\,dx$

Integrate by parts twice to get the same integral on both sides,

$$\begin{array}{ccc} + & \sin(2x) & \mathrm{e}^{-x} \\ - & 2\cos(2x) & -\mathrm{e}^{-x} \\ + & -4\sin(2x) & \mathrm{e}^{-x} \end{array}$$

Therefore, we have:

$$\int \mathrm{e}^{-x}\sin(2x)\,dx = -\mathrm{e}^{-x}(\sin(2x) + 2\cos(x)) - 4\int \mathrm{e}^{-x}\sin(2x)\,dx$$

and

$$\int \mathrm{e}^{-x}\sin(2x)\,dx = -\frac{1}{5}\mathrm{e}^{-x}(\sin(2x) + 2\cos(x)) + C$$

(s) $\displaystyle\int \frac{w}{\sqrt{w+5}}\,dw$

SOLUTION: After some trial and error, we might take

$$u = \sqrt{w+5}$$

We'll need to solve this for w and dw to make the substitution:

$$w = u^2 - 5 \quad \Rightarrow \quad dw = 2u\,du$$

Therefore,

$$\int \frac{w}{\sqrt{w+5}}\,dw = \int \frac{(u^2-5)2u\,du}{u} = \frac{2}{3}u^3 - 10u + C =$$

$$\frac{2}{3}(w+5)^{3/2} - 10(w+5)^{1/2} + C$$

(t) $\displaystyle\int y^2\mathrm{e}^{-3y}\,dy$

SOLUTION: Integration by parts using a table

$$\begin{array}{c|c|c} + & y^2 & \mathrm{e}^{-3y} \\ - & 2y & (-1/3)\mathrm{e}^{-3y} \\ + & 2 & (1/9)\mathrm{e}^{-3y} \\ - & 0 & (-1/27)\mathrm{e}^{-3y} \end{array}$$

Then just write out the answer. Notice that we can factor out $-\mathrm{e}^{-3y}$ to get:

$$-\mathrm{e}^{-3y}\left(\frac{1}{3}y^2 + \frac{2}{9}y + \frac{2}{27}\right) + C$$

(u) $\int \frac{y^3+y}{y+1}\,dy$

Perform long division to get:

$$\int y^2 - y + 2 - \frac{2}{y+1}\,dy = \frac{1}{3}y^3 - \frac{1}{2}y^2 + 2y - 2\ln|y+1| + C$$

(v) (Very similar to part (t) above- Sorry for the duplication!) $\int x^2 e^{2x}\,dx$

SOLUTION: Integration by parts using a table

$$\begin{array}{c|c|c} + & x^2 & e^{2x} \\ - & 2x & (1/2)e^{2x} \\ + & 2 & (1/4)e^{2x} \\ - & 0 & (1/8)e^{2x} \end{array} \qquad \Rightarrow \qquad e^{2x}\left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4}\right) + C$$

(w) $\int (ln(x))^2\,dx$

The integrand itself is hard to integrate, but its derivative might be easier- That's using integration by parts:

$$\begin{array}{ccc} + & (\ln(x))^2 & 1 \\ - & 2\ln(x)/x & x \end{array} \qquad \Rightarrow \quad = x(\ln(x))^2 - 2\int \ln(x)\,dx$$

We can perform integration by parts to get the last integral, or you might just recall what it is:

$$x(\ln(x))^2 - 2(x\ln(x) - x) + C$$

(x)

$$\int \frac{2x^3 - x^2 - 4x - 13}{x^2 - x - 2}\,dx$$

SOLUTION: Do long division first:

$$\frac{2x^3 - x^2 - 4x - 13}{x^2 - x - 2} = 2x + 1 + \frac{x - 11}{x^2 - x - 2} = 2x + 1 + \frac{x - 11}{(x+1)(x-2)}$$

Now expand the last term:

$$\frac{x-11}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

Solve for A, B: $x - 11 = A(x-2) + B(x+1)$. If we substitute $x = -1$, we get $-12 = -3A$, or $A = 4$. If we substitute $x = 2$, we get $-9 = 3B$, or $B = -3$. Therefore,

$$\frac{2x^3 - x^2 - 4x - 13}{x^2 - x - 2} = 2x + 1 + 4\,\frac{1}{x+1} - 3\,\frac{1}{x-2}$$

and the integral is

$$x^2 + x + 4\ln|x+1| - 3\ln|x-2| + C$$