Exam 2 Review

The exam will run approximately 50 minutes. I will give you the sum formulas for $\sum i$ and $\sum i^2$. Other than that, no notes or calculators will be allowed.

Recall that we have Section 11.10 left over after Exam 1. The other topics were sections 4.9 (antiderivatives), 5.1-5.5, then 6.1-6.3.

Section 11.10 topics

- Know how to define Maclaurin and Taylor series for a function $f(x)$ (Equations 6, 7 on p 754).
- Be able to compute a Maclaurin or Taylor series (Exercises 5-9, 13-20).
- Know the template Maclaurin series: $\frac{1}{1-x}$, e^x, and $\sin(x), \cos(x)$. The other table entries you do not need to have memorized (we'll work with them later).
- Use the template series to build new series (exercises 29-33), and to find some sums (exercises 63-68).
- I will not ask you about the remainder R_n for the time being.

Section 4.9

- Know the properties of the antiderivative and some basic antiderivatives (Table 2, p 345). The exceptions are the hyperbolic sine/cosine, which we have not covered $(\sinh(x), \cosh(x))$.
- Be able to find the antiderivative both algebraically (ex 1-22, 25-48) and graphically (51-54).
- Be able to solve some physics problems using acceleration, velocity, displacement, and distance. If you need the acceleration due to gravity (9.8 m/s^2), I will provide that. (Examples 6, 7. Exercises 59-65, 69).

Appendix E, Proof by Induction

Understand how to prove a statement using induction.

Sections 5.1, 5.2

- Know the definition of the definite integral, and how to compute it using right end-points:
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty} \sum_{n=1}^{\infty} f\left(a + i\frac{b-a}{n}\right) \frac{b-a}{n}$$
  This defines the definite integral as "the area under a curve" (or net area, if f is sometimes negative).

- For what functions are we guaranteed that the definite integral exists (as a limit)? If f is continuous, or has only a finite number of jump discontinuities. (This is Theorem 3, p 373).
- Properties 1-8 of the Integral (starts p 379). The two middle equations in the middle of p 379: I won't ask you these specifically, but you should be able to use these in evaluating the integral.
- Be able to write the Riemann sum for a definite integral, then evaluate the sum and limit (ex 21-25, 27).
- Given a Riemann sum, be able to convert it to a definite integral (ex 17-20).
- Evaluate the definite integral graphically (ex 33-34, 51-52, 53).

Section 5.3, 5.4, 5.5

- Know the Fundamental Theorem of Calculus, both parts. Be able to apply the FTC to evaluate the derivative, and to evaluate definite integrals.
- Understand the difference in notation:

$$\int f(x)\,dx \qquad \int_a^x f(t)\,dt \qquad \int_a^b f(x)\,dx$$

- The integral as antiderivative (the indefinite integral, the FTC)
  - Integrate using the table (that we've memorized)
  - Simplify first, then integrate.
  - u, du substitution.
- Applications of the integral:
  - Area between curves
  - Volumes of objects with known cross-sections (like squares or triangles)
  - Volumes of solids of revolution. Use disks, washers and/or shells.

For the sake of time, often on exam questions referring to area, I will ask you to *set up, but do not evaluate* the integral(s) you would need to compute to find the area of a given region (graphs of things more complicated than a parabola or line would be provided). Be sure you're reading the question carefully so that you don't spend a lot of time doing unnecessary computations.

Review Questions

The following questions are not meant to be exhaustive, so you should also be sure you've looked over your old quizzes and understand the homework.

1. State the Fundamental Theorem of Calculus:
2. Give the *definition* of the definite integral $\int_a^b f(x)\,dx =$
3. Find the area bounded between the regions $y = 1 - 2x^2$ and $y = |x|$.
4. For each of the following integrals, write the definition using the Riemann sum, and then evaluate them (MUST use the limit of the Riemann sum for credit, and do not re-write them using the properties of the integral). Note that:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

   (a) $\int_2^5 x^2\,dx$
   (b) $\int_1^3 1 - 3x\,dx$
   (c) $\int_0^5 1 + 2x^3\,dx$

5. Evaluate the integral and interpret it as the area of a region (sketch it).

$$\int_0^4 |\sqrt{x+2} - x|\,dx$$

6. True or False (and give a short reason):

   (a) $\int_0^2 (x - x^3)\,dx$ represents the area under the curve $y = x - x^3$ from 0 to 2.
   (b) If $3 \leq f(x) \leq 5$ for all x, then $6 \leq \int_1^3 f(x)\,dx \leq 10$
   (c) If f, g are continuous on $[a, b]$, then

$$\int_a^b f(x) - g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

   (d) The fact that f, g were each individually continuous on $[a, b]$ was an important thing to state in the last problem.
   (e) If f, g are continuous on $[a, b]$, then

$$\int_a^b f(x)g(x)\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$

   (f) All continuous functions have derivatives.

(g) All continuous functions have antiderivatives.

7. For each of the following Riemann sums, evaluate the limit by first recognizing it as an appropriate integral:

   (a) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{3}{n}\right)\sqrt{1+\frac{3i}{n}}$ (Find four different integrals for this one!)

   (b) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(2+3\cdot\frac{25i^2}{n^2}\right)\left(\frac{5}{n}\right)$

8. Evaluate the integral, if it exists

   (a) $\displaystyle\int_1^9 \frac{\sqrt{u}-2u^2}{u}\,du$

   (b) $\displaystyle\int 3^x+\frac{1}{x}+\sec^2(x)\,dx$

   (c) $\displaystyle\int_{-\pi/4}^{\pi/4}\frac{t^4\tan(t)}{2+\cos(t)}\,dt$

   (d) $\displaystyle\int_0^3 |x^2-4|\,dx$

   (e) $\displaystyle\int \frac{\cos(\ln(x))}{x}\,dx$

   (f) $\displaystyle\int_0^2\sqrt{4-x^2}\,dx$

   (g) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$

   (h) $\displaystyle\int_{-1}^{2}\frac{1}{x}\,dx$

   (i) $\displaystyle\int_0^1(\sqrt[4]{w}+1)^2\,dw$

   (j) $\displaystyle\int_{-2}^{-1}\frac{1}{x}\,dx$

   (k) $\displaystyle\int_0^{1/2}\frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dt$

   (l) $\displaystyle\int(1+\tan(t))\sec^2(t)\,dt$

   (m) $\displaystyle\int\tan(x)\,dx$

   (n) $\displaystyle\int x\sqrt{1+x}\,dx$

   (o) $\displaystyle\int\frac{y-1}{\sqrt{3y^2-6y+4}}\,dy$

   (p) $\displaystyle\int_{-1}^{4}|t-3|\,dt$

9. Find the derivative of the function:

   (a) $\displaystyle F(x)=\int_0^{x^2}\frac{\sqrt{t}}{1+t^2}\,dt$

   (b) $\displaystyle y=\int_{\sqrt{x}}^{3x}\frac{e^t}{t}\,dt$

10. Evaluate by recognizing this as a Riemann Sum:

$$\lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^9+\left(\frac{2}{n}\right)^9+\left(\frac{3}{n}\right)^9+\cdots\left(\frac{n}{n}\right)^9\right]$$

11. Evaluate:

   (a) $\displaystyle\int_0^1\frac{d}{dx}\left(e^{\tan^{-1}(x)}\right)dx$

   (b) $\displaystyle\frac{d}{dx}\int_0^1 e^{\tan^{-1}(x)}\,dx$

   (c) $\displaystyle\frac{d}{dx}\int_0^x e^{\tan^{-1}(t)}\,dt$

12. (a) Sketch the graph of $f(x) = |x| - 1$.
    (b) Suppose this function is the derivative of some other function, $F(x)$. Sketch one possibility using your previous graph as a guide.
    (c) Sketch the function $G(x) = \int_{-2}^{x} f(t)\,dt$ for the same values of $-2 \le x \le 4$, again using your previous answers as a guide.
    (d) What is the relationship (if any) between F and G?

13. A particle moves along a line with velocity $v(t) = t^2 - t$, where v is measured in meters per second. Find (a) the displacement and (b) the distance traveled by the particle during the time interval $[0, 5]$.

14. If f is continuous and $\displaystyle\int_0^9 f(x)\,dx = 4$, find $\displaystyle\int_0^3 x f(x^2)\,dx$

15. If $f''(x) = 2 - 12x$, $f(0) = 0$ and $f(2) = 15$, find $f(x)$.

16. Let R be the region in the first quadrant bounded by $y = x^3$ and $y = 2x - x^2$. Calculate the following quantities: (Exam note: Region R would typically be plotted for you).

    (a) The area of R.
    (b) Volume obtained by rotating R about the $x-$axis.
    (c) Volume obtained by rotating R about the $y-$axis.

17. Use any method to find an integral representing the volume generated by rotating the given region about the specified axis. You do NOT need to evaluate the integral:

    (a) $y = \sqrt{x}, y = 0, x = 1$; about $x = 2$.
    (b) $y = x^2$, $y = 2 - x^2$; about $x = 1$.
    (c) $y = x^2$, $y = 2 - x^2$; about $y = -3$.
    (d) $y = \tan(x)$, $y = x$, $x = \pi/3$; about the $y-$axis.

18. Prove the following using induction:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

19. Prove the following using induction:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

20. Use a series to evaluate the following limit: $\displaystyle\lim_{x \to 0} \frac{\sin(x) - x}{x^3}$

21. Use a known template series to find a series for the following:

(a) $\frac{x^2}{1+x}$ (b) 10^x (c) xe^{2x}

Hint for 5(b): $10^x = e^{\ln(10^x)} = e^{x\ln(10)}$

22. Find the Taylor series for $f(x)$ centered at the given base point:

    (a) $x^4 - 3x^2 + 1$, at $x = 1$

    (b) $1/\sqrt{x}$ at $x = 9$ (just get the first four non-zero terms of the power series).

    (c) x^{-2} at $x = 1$. In this case, find a pattern for the n^{th} coefficient so that you can write the general series. Using this answer, find the radius of convergence.

23. Find the Maclaurin series and radius of convergence for $\ln(x + 1)$.