CALC II FINAL REVIEW SOLUTIONS

1. (6.2, 32) Set up an integral for the volume of the solid obtained by rotating the region defined by $y = \sqrt{x-1}$, $y = 0$ and $x = 5$ about the $y-$axis.

$$V = \int_0^2 \pi(5^2 - (y^2+1)^2)\, dy$$

2. (5.1, 16) Write the area under $y = \sqrt[3]{x}$, $0 \le x \le 8$ as the limit of a Riemann sum (use right endpoints).

$$\Delta x = \frac{8}{n},\ \text{Height}_i = \sqrt[3]{\frac{8i}{n}} \Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt[3]{\frac{8i}{n}} \cdot \frac{8}{n}$$

3. (6.2, 14) Find the volume of the solid obtained by rotating the region bounded by: $y = \frac{1}{x}$, $y = 0$, $x = 1$, $x = 3$ about $y = -1$.

$$V = \int_1^3 \pi \left[\left(\frac{1}{x} + 1\right)^2 - 1^2 \right] dx$$

4. (6.2, 40) The integral $\pi \int_2^5 y\, dy$ represented the volume of a solid. Describe the solid.

Rotate the region to the left of $\sqrt{y}$ and to the right of the $y - axis$ about the $y-$axis ($2 \le y \le 5$).

5. Write the appropriate partial fraction expansion for the following expression (do not solve for the constants):

$$\frac{1+x}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

6. (Ch 6 Review, 16) Let R be the region in the first quadrant bounded by $y = x^3$ and $y = 2x - x^2$. Calculate: (a) The area of R (5/12) (b) Volume obtained by rotating R about the $x-$axis ($41\pi/105$) (c) Volume obtained by rotating R about the $y-$axis ($13\pi/30$), (d) The centroid of R.

7. (5.1, 18) Determine a region whose area is equal to the following limit (do not evaluate the limit): $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1 + \frac{3i}{n}}$

The region is the area under the curve $y = \sqrt{x}$ for $1 \le x \le 4$. You could also say that its under the curve $y = \sqrt{1+x}$ for $0 \le x \le 3$.

8. The derivative of $\sin^{-1}(x)$ is $\frac{1}{\sqrt{1-x^2}}$, the derivative of $\tan^{-1}(x)$ is $\frac{1}{1+x^2}$. To obtain the antiderivatives, we need to use integration by parts:

$$\int \sin^{-1}(x)\, dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\, dx$$

so that

$$\int \sin^{-1}(x)\, dx = x\sin^{-1}(x) + \sqrt{1-x^2}$$

and

$$\int \tan^{-1}(x)\, dx = x\tan^{-1}(x) - \int \frac{x}{1+x^2}\, dx$$

so that

$$\int \tan^{-1}(x)\, dx = x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2)$$

9. Recall that $\frac{d}{dx}\mathrm{e}^{f(x)} = f'(x)\mathrm{e}^{f(x)}$, so the derivative of e^{-2x} is $-2\mathrm{e}^{-2x}$. The antiderivative is $\frac{-1}{2}\mathrm{e}^{-2x}$. The derivative of $\sin(3x)$ is $3\cos(3x)$, the antiderivative is $-\frac{1}{3}\cos(3x)$.

10. Suppose you are integrating $P(x)/Q(x)$, where P and Q are polynomials. Explain the process by which we integrate this expression. (Consider the degrees of P and Q). First, if the degree of $P \ge$ the degree of Q, perform long division. We now can assume that the degree of P is less than the degree of Q. Factor Q completely, and use partial fractions.

11. What was the Mean Value Theorem for Integrals? The same as the average value formula. If f is continuous on the interval $[a, b]$, then there is a c in the interval such that:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\, dx$$

12. Evaluate: $\int_0^\infty t\mathrm{e}^{-st}\, dt$, where s is a positive constant. Integration by parts (use a table) gives:

$$\lim_{T\to\infty} \frac{-t}{s}\mathrm{e}^{-st}\Big|_0^T + \lim_{T\to\infty} \frac{-1}{s^2}\mathrm{e}^{-st}\Big|_0^T$$

Use L'Hospital's rule to compute the limits to get that the first term is 0, and the overall result is $1/s^2$.

13. Write the following limit as a definite integral: on the given interval: $\lim_{n\to\infty}\sum_{i=1}^{n}\left(2 + \frac{\pi i}{n}\right)\sin\left(2 + \frac{\pi i}{n}\right)\cdot\frac{\pi}{n}$. (NOTE THE MISPRINT)

We see, from the form $\sum f\left(a + \frac{b-a}{n}i\right)\cdot\frac{b-a}{n}$, that $a = 2$, $f(x) = x\sin(x)$, and so $b - a$ is π. Therefore, the integral is:

$$\int_2^{2+\pi} x\sin(x)\, dx$$

14. If $f(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$,

(a) Find the interval on which the sum converges.

This is a geometric series with $r = x$. Therefore, the series converges absolutely if $|x| < 1$, and diverges if $x = 1$ or $x = -1$.

(b) Viewing this as a geometric series, what is the formula for f?

From the sum of a geometric series, $f(x) = \frac{1}{1-x}$

(c) Find $f'(x)$ by differentiating the power series. If you also differentiate your answer to part (b), then you'll get a formula for the new power series.

Differentiating the power series, we get:

$$f'(x) = \sum_{n=1}^{\infty} nx^{n-1}$$

Differentiating part (b), we get:

$$f'(x) = \frac{-1}{(1-x)^2}$$

so:

$$\frac{-1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}, |x| < 1$$

(d) Find $\int f(x)\, dx$ by integrating the power series. What is the formula for the resulting sum (using the antiderivative of part (b))?

Similar to what we just did,

$$\int f(x)\, dx = \sum_{n=0}^{\infty} \frac{1}{n+1}x^{n+1} + C$$

and

$$\int \frac{1}{1-x}\, dx = \ln|1-x| + C$$

Setting the two equal at $x = 0$, we see that the constant is zero, so

$$\ln|1-x| = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

Side Note: From this formula, we can sum the alternating harmonic series. That is, if $x = -1$,

$$\ln(2) = \ln|1-(-1)| = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

15. (6.1, 17) Find the area between the curves $y^2 = x$ and $x - 2y = 3$.
Taking horizontal rectangles (integrate with respect to y), we see that the rightmost function is $x - 2y = 3$. The points of intersection are $(1,-1)$ and $(9,3)$. The area is:
$$\int_{-1}^{3} (2y+3) - y^2\ dy = \frac{32}{3}$$

16. (5.2, 44) Write the following difference as a single integral: $\int_2^{10} f(x)\ dx - \int_2^7 f(x)\ dx$
$$\int_7^{10} f(x)\ dx$$

17. (5.2, 46) If $\int_0^1 f(x)dx = 2, \int_0^4 f(x)dx = -6$, and $\int_3^4 f(x)\ dx = 1$, find $\int_1^3 f(x)\ dx$. $-6-2-1=-9$

18. (Similar to 5.2, 39) If $\int_0^1 f(x)\ dx = \frac{1}{3}$, what is $\int_0^1 5 - 6f(x)\ dx$? $5(1) - 6\frac{1}{3} = 5-2 = 3$.

19. (5.3, 9) Compute $\frac{dF}{dx}$, if $F(x) = \int_x^2 \cos(t^2)\ dt$ Note first that $\int_x^2 \cos(t^2)dt = -\int_2^x \cos(t^2)\ dt$, which is now in standard form.
$$-\cos(x^2)$$

20. (5.3, 13) Compute $\frac{dg}{dy}$, if $g(y) = \int_3^{\sqrt{y}} \frac{\cos(t)}{t}\ dt$.
$$\frac{cos(\sqrt{y})}{\sqrt{y}} \cdot \frac{1}{2\sqrt{y}}$$

21. (5.3, 50) Find $\frac{dy}{dx}$, if $y = \int_{\cos(x)}^{5x} cos(t^2)\ dt$
First, in standard form, (0 was convenient, use any constant):
$$y = -\int_0^{\cos(x)} \cos(t^2)\ dt + \int_0^{5x} \cos(t^2)\ dt$$
so the derivative is:
$$\frac{dy}{dx} = \cos(\cos^2(x)) \cdot \sin(x) + \cos(25x^2) \cdot 5$$

22. (Section 4.4, L'Hospital) If $a_n = \left(\frac{n+1}{n}\right)^n$, then
$$a_n = \mathrm{e}^{n\ln((n+1)/n)}$$
so we'll take the limit of the exponent:
$$\lim_{n\to\infty} n\ln((n+1)/n) = \lim_{n\to\infty} \frac{\ln((n+1)/n)}{1/n} =$$
$$\lim_{n\to\infty} \frac{\ln(n+1)-\ln(n)}{1/n} = \lim_{n\to\infty} \frac{\frac{1}{n+1} - \frac{1}{n}}{-1/n^2} =$$
$$\lim_{n\to\infty} \frac{n^2}{n(n+1)} = 1$$
so overall, $\lim_{n\to\infty} a_n = \mathrm{e}^1 = \mathrm{e}$ Thus, the series converges to e.
If we were considering $\sum_{n=1}^{\infty} a_n$, then the divergence test would say that this sum diverges.

23. (5.3, 58) Evaluate the limit by recognizing the sum as a Riemann sum:
$$\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \ldots + \sqrt{\frac{n}{n}}\right)$$
$$\int_1^4 \frac{1}{\sqrt{x}}\ dx = 2$$

24. (5.2, 22) Write the following integral as the limit of a Riemann sum (use right endpoints): $\int_0^5 (1+2x^3)\ dx$
$$\lim_{n\to\infty} \sum i = 1^n \left(1 + \left(\frac{5i}{n}\right)^3\right)\frac{5}{n}$$

25. (5.2, 37) Given that $\int_4^9 \sqrt{x}\ dx = \frac{38}{3}$, what is $\int_9^4 \sqrt{t}\ dt$? $-38/3$

26. (6.5, 10) Let $f(x) = \mathrm{e}^x$ on the interval $[0,2]$. (a) Find the average value of f. (b) Find c such that $f_{\text{avg}} = f(c)$.
$$f_{\text{avg}} = \frac{1}{2}\int_0^2 \mathrm{e}^x\ dx = \frac{1}{2}(\mathrm{e}^2 - 1)$$
$$\mathrm{e}^c = \frac{\mathrm{e}^2-1}{2} \Rightarrow c = \ln\left(\frac{\mathrm{e}^2-1}{2}\right)$$

27. (5.4, 53) The velocity function is $v(t) = 3t - 5, 0 \le t \le 3$ (a) Find the displacement, (b) Find the distance travelled. Displacement is $\int_0^3 3t - 5\ dt = -3/2$ Distance is $\int_0^3 |3t - 5|\ dt = \int_0^{5/3} -3t + 5\,dt + \int_{5/3}^3 3t - 5\,dt = \frac{25}{6} + \frac{8}{3} = \frac{41}{6}$

28. Exercise 7, pg. 427 (There are some graphs to consider). See the back of the book.

29. (Ch 5 Review, 68) Suppose $h(1) = -2$, $h'(1) = 2$, $h''(1) = 3$, $h(2) = 6$, $h'(2) = 5$, and $h''(2) = 13$, and h'' is continuous. Evaluate $\int_1^2 h''(u)\ du$. $h'(2) - h'(1) = 5 - 2 = 3$.

30. (6.1, 24) Find the area between the curves $y = |x|$ and $y = x^2 - 2$.
$$\int_{-2}^{2} |x| - (x^2-2)\ dx = 2\int_0^2 x - (x^2-2)\ dx = \frac{20}{3}$$

31. (5.5, 78) If f is continuous and $\int_0^9 f(x)\ dx = 4$, find $\int_0^3 xf(x^2)\ dx$. Let $u = x^2$, so $du = 2x\ dx$ Then:
$$\int_0^3 xf(x^2)\ dx = \frac{1}{2}\int_0^9 f(u)\ du = 2.$$

32. For each function, find the Taylor series for $f(x)$ centered at the given value of a:
    (a) $f(x) = 1 + x + x^2$ at $a = 2$
    We need to compute $f(2), f'(2), f''(2)$, which are $7, 5, 2$ respectively. Now,
    $$f(x) = 7 + 5(x-2) + \frac{2}{2}(x-2)^2$$
    You can check that this simplifies to the original.
    (b) $f(x) = \mathrm{e}^x$ at $a = 3$.
    We need to compute $f(3), f'(3), f''(3), \ldots$. Since the derivative of e^x is e^x, $f^{(n)}(3) = \mathrm{e}^3$ for all n, and:
    $$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{\mathrm{e}^3}{n!}(x-3)^n$$

(c) $f(x) = \frac{1}{x}$ at $a = 1$.
We start computing derivatives:

$$f(x) = x^{-1}, f'(x) = -x^{-2}, f''(x) = 2x^{-3},$$

$$f'''(x) = -3 \cdot 2x^{-4}, f^{(iv)}(x) = 4 \cdot 3 \cdot 2x^{-5}$$

so the formula is: $f^{(n)}(x) = (-1)^n n! x^{-(n+1)} \Rightarrow f^{(n)}(1) = (-1)^n n!$ Therefore, we have the Taylor expansion:

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!}(x-1)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

which we can see immediately is valid for $|x - 1| < 1 \Rightarrow 0 < x < 2$, which we might expect since $\frac{1}{x}$ is not defined at $x = 0$.

33. Problem 41, p. 439 (see the book).

34. (6.1, 46(a)) Find a so that half the area under the curve $y = \frac{1}{x^2}$ lies in the interval $[1, a]$ and half of the area lies in the interval $[a, 4]$.

$$\int_1^4 \frac{1}{x^2}\,dx = 2\int_1^a \frac{1}{x^2}\,dx$$

$$1 - \frac{1}{4} = 2 - \frac{2}{a} \Rightarrow a = \frac{8}{5}$$

35. (6.3, 22) Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by $y = x$, $y = 4x - x^2$, about $x = 7$. (Shells)

$$\int_0^3 2\pi(7 - x)((4x - x^2) - x)\,dx$$

36. Compute:

(a) $\displaystyle \frac{d}{dx}\int_{3x}^{\sin(x)} t^3\,dt$
This is the Fundamental Theorem of Calculus, Part I. Note that we can write this as:

$$-\int_0^{3x} t^3\,dt + \int_0^{\sin(x)} t^3\,dt$$

choosing zero at random. This can also be further written as:

$$-g(3x) + g(\sin(x))$$

where $g(x) = \displaystyle\int_0^x t^3\,dt$, and $g'(x) = x^3$. Now, the derivative is found via the chain rule:

$$-g'(3x) \cdot 3 + g'(\sin(x)) \cdot \cos(x)$$

which gives:

$$-(3x)^3 \cdot 3 + (\sin(x))^3 \cos(x) = -81x^3 + \sin^3(x)\cos(x)$$

(b) $\displaystyle \frac{d}{dx}\int_1^5 x^3\,dx$
The derivative is zero- we're differentiating a constant.

(c) $\displaystyle \int_1^5 \frac{d}{dx} x^3\,dx$

This will be $\displaystyle\int_1^5 3x^2\,dx = x^3\Big|_1^5 = 5^3 - 1 = 124$

37. Define the integral of f:

(a) if f is continuous on $[a, b]$ (as a Riemann Sum)

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(a+i((b-a)/n)) \cdot (b-a)/n$$

(b) On the interval $[a, b]$ if f has a vertical asymptote at $x = a$, but is continuous on $(a, b]$.

$$\int_a^b f(x)\,dx = \lim_{T\to a+} \int_T^b f(x)\,dx$$

if this limit exists.

(c) On the interval $[a, \infty)$, if f is continuous there.

$$\int_a^\infty f(x)\,dx = \lim_{T\to\infty} \int_a^T f(x)\,dx$$

if this limit exists.

38. What is the difference between a sequence, a series, and a power series?
A sequence is a listing of numbers, a series is a sum of numbers, and a power series is a function. Our notation changes from a_n to $\sum a_n$ to $\sum c_n(x-a)^n$

39. What does it mean for a infinite series to "converge"? Be specific in your answer using s_n as the sum for $k = 1$ to n.
Let $s_n = \sum_{k=1}^n a_k$. Then the infinite series is said to converge if the $\lim_{n\to\infty} s_n$ exists. Note that we never use this to prove that a series converges- we have our tests on a_k for that... In that sense, this is a theoretical, rather than practical, result.

40. What does it mean (graphically) for a sequence to "converge"?
The set of plotted points, (n, a_n) converges if there is a horizontal asymptote.

41. If $a_n = \frac{n!}{(n+2)!}$, does the sequence converge? If so, to what does it converge?
The sequence does converge: $a_n = \frac{1}{(n+1)(n+2)}$, so the limit as $n \to \infty$ is zero.

42. If we think of a_n as $f(n)$, then what is the relationship between $\sum_{n=1}^T a_n$ and $\int_1^T f(x)\,dx$? You may assume f is decreasing and positive. Hint: There are two possibilities, where you use either right endpoints or left endpoints in a Riemann sum.
By using right endpoints,

$$\sum_{n=2}^T a_n < \int_1^T f(x)\,dx$$

By using left endpoints,

$$\sum_{n=1}^{T-1} a_n > \int_1^T f(x)\,dx$$

43. The function $f(x)$ is given as straight lines going through the points $(0, 1)$, $(2, 3)$, and $(5, 0)$. Compute $\int_0^5 f(x)\,dx$ using geometry.
One way is to use a rectangle and two triangles:

$$(1)(2) + \frac{1}{2}(2)(2) + \frac{1}{2}(3)(3) = \frac{17}{2}$$

SERIES solutions

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$

We see that this series is a lot like $\sum \frac{1}{n}$, which diverges. We use the limit comparison test:

$$\lim_{n\to\infty} \frac{n-1}{n^2+n}\cdot\frac{n}{1} = \lim_{n\to\infty}\frac{n^2-n}{n^2+n} = \lim_{n\to\infty}\frac{1-\frac{1}{n}}{1+\frac{1}{n}} = 1$$

Therefore, the series will diverge by the limit comparison test with the harmonic series.

2. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n}{1+8n}\right)^n$

Use the root test:

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{3n}{1+8n}\right)^n} = \lim_{n\to\infty}\frac{3n}{1+8n} = \frac{3}{8}$$

The series converges (absolutely) by the root test with the limit of $\frac{3}{8} < 1$.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n!}$

Use the ratio test:

$$\lim_{n\to\infty}\frac{10^{n+1}}{(n+1)!}\cdot\frac{n!}{10^n} = \lim_{n\to\infty}\frac{10}{n+1} = 0$$

The series converges (absolutely) by the ratio test with the limit $0 < 1$.

4. $\displaystyle\sum_{n=1}^{\infty} n^2 \mathrm{e}^{-n^3}$

This looks set up for the integral test:

$$\int_1^{\infty} x^2\mathrm{e}^{-x^3}\,dx$$

with $u = x^3$, $du = 2x^2\,dx$, we get:

$$\frac{1}{2}\int \mathrm{e}^{-u}\,du = -\frac{1}{2}\mathrm{e}^{-x^3}\Big|_1^T = -\frac{1}{2}\left(\mathrm{e}^{-T^3} - \mathrm{e}^{-1}\right)$$

and the limit as $T\to\infty$ is $\frac{1}{2}\mathrm{e}^{-1}$. Therefore, the integral converges, so the associated series will also converge (absolutely).

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

We see that this is not going to be absolutely convergent (formally, we can do a limit comparison with the $p-$series $\frac{1}{\sqrt{n}}$. We will check for conditional convergence via the Alternating Series Test. We need to show that the sequence of terms is decreasing:

$$f(x) = \frac{1}{\sqrt{x}-1} \Rightarrow f'(x) = \frac{-1}{(\sqrt{x}-1)^2}\cdot\frac{1}{2\sqrt{x}}$$

so the derivative is negative for $x > 0$. We also see that:

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}-1} = 0$$

so this series converges (conditionally) by the Alternating Series Test.

6. $\displaystyle\sum_{k=1}^{\infty} \frac{k+5}{5^k}$

Using the ratio test:

$$\lim_{k\to\infty}\frac{(k+1)+5}{5^{n+1}}\cdot\frac{5^n}{k+5} = \lim_{k\to\infty}\frac{k+6}{5^{n+1}}\cdot\frac{5^n}{k+5} =$$

$$\lim_{k\to\infty}\frac{1}{5}\cdot\frac{k+6}{k+5} = \frac{1}{5}$$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n+1)(n+2)}$

Intuitively, this series will behave like the alternating harmonic series $\sum\frac{(-1)^n}{n}$. Let's check:

We will show that the series of positive terms will diverge by using the limit comparison with $\sum\frac{1}{n}$:

$$\lim_{n\to\infty}\frac{n}{(n+1)(n+2)}\cdot\frac{n}{1} = 1$$

Now we need to show the series satisfies the hypotheses of the Alternating Series Test. First, we show the sequence of positive terms is decreasing, then the limit of those terms is zero:

$$f(x) = \frac{x}{(x+1)(x+2)} \Rightarrow f'(x) = \frac{-x^2+4x+2}{((x+1)(x+2))^2}$$

The numerator is going to negative infinity as $x\to\infty$, and the denominator is always positive. Therefore, for x large enough, $f'(x)$ will always be negative. Specifically, that value of x can be found by using the quadratic formula:

$$x = \frac{-4\pm\sqrt{16+8}}{-2} \approx 2\pm 2.45$$

so for $x > 5$ (next integer up) , $f'(x) < 0$. We also have that:

$$\lim_{n\to\infty}\frac{n}{n^2+3n+2} = \lim_{n\to\infty}\frac{1}{n+3+(2/n)} = 0$$

By the Alternating Series Test, the series will converge (conditionally).

8. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$

This series will behave like $\sqrt{n^2}/n^3 = 1/n^2$, so we expect convergence. We'll use the limit comparison test:

$$\lim_{n\to\infty}\frac{\sqrt{n^2-1}}{n^3+2n^2+5}\cdot\frac{n^2}{1} = \lim_{n\to\infty}\frac{n^2\sqrt{n^2-1}}{n^3+2n^2+5} =$$

$$\lim_{n\to\infty}\frac{(n^2\sqrt{n^2-1})/n^3}{(n^3+2n^2+5)/n^3} = \lim_{n\to\infty}\frac{\frac{\sqrt{n^2-1}}{n}}{1+\frac{2}{n}+\frac{5}{n^3}} =$$

$$\lim_{n\to\infty}\frac{\sqrt{\frac{n^2-1}{n^2}}}{1+\frac{2}{n}+\frac{5}{n^3}} = \lim_{n\to\infty}\frac{\sqrt{1-\frac{1}{n^2}}}{1+\frac{2}{n}+\frac{5}{n^3}} = 1$$

This shows that indeed, the unknown series converges like $\sum\frac{1}{n^2}$. Thus, by the Limit Comparison Test, the given series converges (absolutely).

9. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n/2)}{n^2+4n}$

We see that $|\cos(n/2)| < 1$ for all n. Therefore, we can directly compare this series to $\sum\frac{1}{n^2+4n}$, which we can show is convergent by the Limit Comparison:

$$\lim_{n\to\infty}\frac{1}{n^2+4n}\cdot\frac{n^2}{1} = 1$$

Therefore, by the Direct Comparison Test, the unknown series converges absolutely (and is therefore convergent).

You could have done the Direct Comparison this way as well:

$$\frac{|\cos(n/2)|}{n^2+4n} \le \frac{1}{n^2+4n} \le \frac{1}{n^2}$$

The last inequality follows because if you reduce the size of the denominator, you increase the size of the fraction (like the difference between 1/10 and 1/2).

10. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$

Probably easiest to use the Ratio Test:

$$\lim_{n\to\infty} \frac{(n+1)^2+1}{5^{n+1}} \cdot \frac{5^n}{n^2+1} = \lim_{n\to\infty} \frac{1}{5} \cdot \frac{(n+1)^2+1}{n^2+1} =$$

$$\lim_{n\to\infty} \frac{1}{5} \cdot \frac{n^2+2n+2}{n^2+1} =$$

$$\lim_{n\to\infty} \frac{1}{5} \cdot \frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}} = \frac{1}{5}$$

The series converges (absolutely) by the Ratio Test, where the limit was $\frac{1}{5} < 1$.

11. $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{\sqrt{k}}{k+5}$

We see that the series will not converge absolutely, since the terms go to zero like $\frac{1}{\sqrt{k}}$, which is a $p-$series with $p < 1$. Therefore, we go directly to the Alternating Series Test to see if we get conditional convergence.

First, check that the (positive) terms of the series are decreasing.

$$f(x) = \frac{\sqrt{x}}{x+5} \Rightarrow f'(x) = \frac{\frac{1}{2\sqrt{x}}(x+5) - \sqrt{x}}{(x+5)^2}$$

which simplifies to:

$$f'(x) = \frac{-x+5}{2\sqrt{x}(x+5)^2}$$

The denominator is always positive for $x > 0$, so the only thing determining the sign of f' is the numerator, which is negative if $x > 5$.

Now, $\lim_{n\to\infty} \frac{\sqrt{n}}{n+5} = \lim_{n\to\infty} \frac{1}{\sqrt{n}+\frac{5}{\sqrt{n}}} = 0$, so by the Alternating Series Test, the series will converge (but only conditionally).

12. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$

This one is ripe for the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n}{n^2}\right)^n} = \lim_{n\to\infty} \frac{2n}{n^2} = \lim_{n\to\infty} \frac{2}{n} = 0$$

By the Root Test with a limit of $0 < 1$, the series will converge (absolutely).

13. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

Whenever we see an $n!$, we almost always want to use the Ratio Test, and this is no exception:

$$\lim_{n\to\infty} \frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2} = \lim_{n\to\infty} \frac{3\cdot(n+1)^2}{(n+1)n^2} =$$

$$\lim_{n\to\infty} 3 \cdot \frac{n+1}{n^2} = 0$$

By the Ratio Test with a limit of $0 < 1$, the series will converge (absolutely).

14. $\displaystyle\sum_{k=1}^{\infty} k^{-1.7}$

Note that:

$$k^{-1.7} = \frac{1}{k^{1.7}} = \left(\frac{1}{k}\right)^{1.7}$$

so this is a $p-$series with $p > 1$. Therefore, the series will converge (absolutely).

Power Series Solutions

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n+1}$

We'll use the Ratio Test to check for convergence:

$$\lim_{n\to\infty} \left|\frac{x^{n+1}}{n+2}\right| \cdot \left|\frac{n+1}{x^n}\right| = \lim_{n\to\infty} |x| \cdot \frac{n+1}{n+2} = |x|$$

The series will converge absolutely when $|x| < 1$ by the Ratio Test. If $x = 1$, the series will be:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$$

which is an Alternating Harmonic Series (converges conditionally). We essentially get the same thing if $x = -1$; the series is multiplied by -1. Our conclusion is that the series will converge if $-1 \le x \le 1$ (so the Radius of Convergence is also 1).

2. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

We'll take the same route as we did previously with the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)^2}\right| \cdot \left|\frac{n^2}{x^n}\right| = \lim_{n\to\infty} |x| \cdot \left(\frac{n+1}{n}\right)^2 = |x|$$

And check the endpoints: In either case, (in absolute value) the terms will be $1/n^2$, which converges by the $p-$series.

The series will converge for $-1 \le x \le 1$.

3. $\displaystyle\sum_{n=1}^{\infty} n^n x^n$

This one is set up for the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|(nx)^n|} = \lim_{n\to\infty} n|x|$$

This term goes to infinity for all $x \neq 0$, so the only value of x for which this series will converge is $x = 0$.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 x^n}{10^n}$

There are some choices with this one, we'll go through two of them:

(a) Use the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n^2 x^n}{10^n}\right|} = \lim_{n\to\infty} n^{2/n} \frac{|x|}{10}$$

We can take the limit of $n^{2/n}$ separately:

$$\lim_{n\to\infty} n^{2/n} = \lim_{n\to\infty} \mathrm{e}^{\frac{2}{n}\ln(n)}$$

Now,

$$\lim_{n\to\infty} \frac{2\ln(n)}{n} = \lim_{n\to\infty} \frac{\frac{2}{n}}{1} = 0$$

so overall, $n^{2/n} \to \mathrm{e}^0 = 1$.

Thus, the overall limit from the Root Test is $\frac{|x|}{10}$, so $|x| < 10$. If $x = 10$, the original series becomes $\sum n^2$ which diverges (same if $x = -10$). So in this case, $-10 < x < 10$.

(b) Use the Ratio Test:

$$\lim_{n\to\infty} \frac{(n+1)^2 |x^{n+1}|}{10^{n+1}} \cdot \frac{10^n}{n^2|x^n|}$$

which simplifies to:

$$\lim_{n\to\infty} \frac{|x|}{10}\left(\frac{n+1}{n}\right)^2 = \frac{|x|}{10}$$

so again we get that $|x| < 10$, and we already considered the cases where $x = 10, x = -10$, so the final answer is that the series converges for $-10 < x < 10$.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}$
Use the Ratio Test:

$$\lim_{n\to\infty} \frac{|3x-2|^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{|3x-2|^n} =$$

$$\lim_{n\to\infty} \frac{|3x-2|}{3} \cdot \left(\frac{n}{n+1}\right) = \frac{|3x-2|}{3}$$

For convergence, $|3x-2| < 3$ or

$$3x - 2 < 3 \Rightarrow 3x < 5 \Rightarrow x < 5/3$$

and

$$-3x + 2 < 3 \Rightarrow 3x > -1 \Rightarrow x > -1/3$$

If $x = 5/3$, the series becomes $\sum \frac{1}{n}$, which diverges, and if $x = -1/3$, the series becomes $\sum(-1)^n \frac{1}{n}$, which converges.
The interval on which the series converges is: $\frac{-1}{3} \le x < \frac{5}{3}$.

6. $\displaystyle\sum_{n=1}^{\infty} n^3(x-5)^n$
By the Ratio Test:

$$\lim_{n\to\infty} \frac{(n+1)^3|x-5|^{n+1}}{n^3|x-5|^n} = |x-5|$$

so $4 < x < 6$. If $x = 4$, the series becomes $\sum(-1)^n n^3$, which diverges, and if $x = 6$, the series becomes $\sum n^3$ which diverges. The interval for which the series will converge is:

$$4 < x < 6$$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}$
By the Ratio Test:

$$\lim_{n\to\infty} \frac{|x|^{2(n+1)-1}}{(2(n+1)-1)!} \cdot \frac{(2n-1)!}{|x|^{2n-1}} = \lim_{n\to\infty} \frac{|x|^2}{(2n)(2n+1)} = 0$$

for all x. This series converges for all x.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^2}$
By the Ratio Test:

$$\lim_{n\to\infty} \frac{|4x+1|^{n+1}}{(n+1)^2} \cdot \frac{n^2}{|4x+1|^n} =$$

$$\lim_{n\to\infty} |4x-1| \cdot \left(\frac{n}{n+1}\right)^2 = |4x-1|$$

so $|4x-1| < 1 \Rightarrow \frac{-1}{2} < x < \frac{1}{2}$. If $x = -1/2$, the sum becomes $\sum \frac{(-1)^n}{n^2}$, which converges absolutely by the $p-$series, and if $x = \frac{1}{2}$, $\sum \frac{1}{n^2}$ converges absolutely. In this case, $\frac{-1}{2} \le x \le \frac{1}{2}$.

9. Evaluate:

(a) (Ch 7 Review, 4) $\displaystyle\int \frac{\sec^2(x)}{1-\tan(x)}\,dx$ Let $u = \tan(x)$. $-\ln|1-\tan(x)| + C$

(b) $\displaystyle\int_0^{\infty} \frac{1}{(x+2)(x+3)}\,dx$
Using partial fractions, the integral is:

$$\int \frac{1}{x+2} - \frac{1}{x+3}\,dx$$

so $\ln|x+2| - \ln|x+3| = \ln\left|\frac{x+2}{x+3}\right| + C$

(c) $\displaystyle\int \ln(x)\,dx$ Integration by parts with $u = \ln(x)$, $dv = dx$:

$$x\ln|x| - x + C$$

(d) (5.5, 31) $\displaystyle\int \frac{dx}{x\ln(x)}$ Let $u = \ln(x)$, so $\ln|\ln|x|| + C$

(e) (Ch 7 Review, 6) $\displaystyle\int \frac{1}{y^2-4y-12}\,dy$ Use partial fractions to get $\frac{1}{8}\ln|y-6| - \frac{1}{8}\ln|y+2| + C$

(f) (5.4, 9) $\displaystyle\int (1-t)(2+t^2)\,dt$ Multiply it out, $2t - t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + C$

(g) (5.4, 25) $\displaystyle\int u(\sqrt{u} + \sqrt[3]{u})\,du$ Simplify before integrating. $\frac{2}{5}u^{5/2} + \frac{3}{7}u^{7/3} + C$

(h) (Ch 7 Review, 33) $\displaystyle\int_1^{\infty} \frac{1}{(2x+1)^3}\,dx$ 1/36.

(i) $\displaystyle\int_0^3 \frac{1}{\sqrt{x}}\,dx$
We'll need a limit:

$$\lim_{T\to 0+} 2x^{1/2}\Big|_T^3 = 2\sqrt{3}$$

(j) (Ch 7 Review, 36) $\displaystyle\int_1^4 \frac{e^{1/x}}{x^2}\,dx$ Let $u = 1/x$ and substitute. $e - e^{1/4}$

(k) $\displaystyle\int e^{-x}\sin(2x)\,dx$
Integration by parts twice so that:

$$\int e^{-x}\sin(2x)\,dx =$$

$$e^{-x}\left(\frac{-1}{2}\cos(2x) - \frac{1}{4}\cos(2x)\right) - \frac{1}{4}\int e^{-x}\sin(2x)\,dx$$

Add the integral back to the left hand side to get a final answer of

$$e^{-x}\left(-\frac{2}{5}\cos(2x) - \frac{1}{5}\sin(2x)\right) + C$$

(l) (Ch 7 Review, 29) $\displaystyle\int \frac{x^2}{(4-x^2)^{3/2}}\,dx$ Let $x = 2\sin(\theta)$ and substitute.

$$\frac{x}{\sqrt{4-x^2}} - \sin^{-1}(x/2) + C$$

(m) (Ch 7 Review, 26) $\displaystyle\int \frac{1}{1+e^x}\,dx$ Let $u = e^x$, so $\ln(u) = x$ and $(1/u)du = dx$ and substitute. You will then need to do partial fractions, and get:

$$x - \ln(1+e^x) + C$$

(n) (5.5, 22) $\displaystyle\int \frac{\tan^{-1}(x)}{1+x^2}\ dx$ Let $u = \tan^{-1}(x)$. $\frac{1}{2}(\tan^{-1}(x))^2 + C$

(o) (5.5, 70) $\displaystyle\int_{-a}^{a} x\sqrt{x^2-a^2}\ dx$ You should get 0.

(p) (5.5, 56 (+section 7.8)) $\displaystyle\int_0^2 \frac{dx}{(2x-3)^2}\ dx$ Does not exist (take the limit as $T \to \frac{3}{2}$).

(q) (Ch 7 Review, 13) $\displaystyle\int \sin^2 x \cos^5 x\ \ dx$ Pull out a $\cos(x)$ to keep with dx.

$$\frac{1}{3}\sin^3(x) - \frac{2}{5}\sin^5(x) + \frac{1}{7}\sin^7(x) + C$$

(r) $\displaystyle\int \frac{\sqrt{9x^2-4}}{x}\ dx$

Let $3x = 2\sec(\theta)$, so $\sqrt{9x^2-4} = 2\tan(\theta)$, and $dx = (2/3)\sec(\theta)\tan(\theta)d\theta$. Substituting and simplifying, we get

$$2\int tan^2(\theta)\, d(\theta) = 2\int sec^2(\theta) - 1\, d(\theta)$$

which is $2\tan(\theta) - 2\theta + C$. Converting back to x using a triangle (hyp=3x, adj=2, opp=$\sqrt{9x^2-4}$)

$$\sqrt{9x^2-4} - 2\cos^{-1}(\frac{2}{3x}) + C$$

(You could have had $\sec^{-1}(3x/2)$ in the last term, too).

(s) (Ch 7 Review, 19) $\displaystyle\int \frac{1}{\sqrt{x^2-4x}}\ dx$ First complete the square then substitute $x - 2 = \sec(\theta)$. You should end up with $\int \sec(\theta)\ d\theta$, which will be given on the exam... Be sure you can go back to x.

(t) (Ch 7 Review, 5) $\displaystyle\int x^4 \ln(x)\ dx$ Integrate by parts with $u = \ln(x)$, $dv = x^4\ dx$.

$$\frac{1}{5}x^5\ln(x) - \frac{1}{25}x^5 + C$$

(u) $\displaystyle\int \frac{2}{3x+1} + \frac{2x+3}{x^2+9}\ dx$ Split into 3 integrals:

$$\int \frac{2}{3x+1}\ dx = \frac{2}{3}\ln|3x+1| + C$$

Next one, use $u = x^2 + 9$

$$\int \frac{2x}{x^2+9}\ dx = \ln(x^2+9)$$

Next one, use the table

$$\int \frac{3}{x^2+9}\ dx = \tan^{-1}(x/3)$$

Final answer: Sum them together and add C.

(v) $\displaystyle\int x^2\cos(3x)\, dx$ Straight integration by parts (Grouped to save space)

$$\left(\frac{x^2}{3} - \frac{2}{27}\right)\sin(3x) + \frac{2x}{9}\cos(3x)$$